Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **46** (2026) no. 1, 113–122. doi:10.1285/i15900932v46n1p113-122

Some results on local Cohomology Modules and Serre subcategories

Shahram Rezaei
Department of Mathematics, Faculty of Science, Payame Noor University (PNU), Tehran, Iran
sha.rezaei@pnu.ac.ir

Received: 16.10.24; accepted: 01.06.26.

Abstract. Let R be a commutative Noetherian ring, $\mathfrak{a}$ an ideal of R and let M be a finitely generated R-module and N be an arbitrary R-module. Let n be an integer and $\mathcal{S}$ be a Serre subcategory of the category of R-modules. We study some properties of local cohomology modules by using the concept of Serre subcategory of the category of R-modules. More precisely, we investigate the membership of $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ and $\mathrm{H}^i_{\mathrm{Ann}_R\,(M/\mathfrak{a}M)}(N)$ in $\mathcal{S}$ for all $i<n$ and as the dual case, we find a relation between $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ and $\mathrm{H}^i_{\mathrm{Ann}_R\,(N/\mathfrak{a}N)}(M,R)$ for all $i>n$.

Keywords: Local cohomology, Serre subcategory

MSC 2020 classification: primary 13D45, secondary 13C99

Introduction

Throughout, R is a commutative Noetherian ring with identity and $\mathfrak{a}$ is an ideal of R. Let M and N be two R-modules with M finitely generated. The i-th generalized local cohomology module of M and N with respect to $\mathfrak{a}$ is defined by

$$\mathrm{H}^i_{\mathfrak{a}}(M,N) := \varinjlim_n \mathrm{Ext}^i_R(M/\mathfrak{a}^n M, N).$$

Recall that a class $\mathcal{S}$ of R-modules is a Serre subcategory of the category of R-modules, when it is closed under taking quotients, submodules and extensions. We say that a Serre subcategory $\mathcal{S}$ satisfies the condition $\mathcal{C}_{\mathfrak{a}}$ if for every $\mathfrak{a}$-torsion R-module M such that $0:_M \mathfrak{a}$ is in $\mathcal{S}$, then M is in $\mathcal{S}$ and we denote it by $\mathcal{S}\in\mathcal{C}_{\mathfrak{a}}$ (See [1, Definition 2.1]). Recall that an R-module M is called $\mathfrak{a}$-torsion if $M=\cup_{n\in\mathbb{N}}(0:_M \mathfrak{a}^n)$.

At first, in [1] the authors studied local cohomology modules by means of Serre subcategories. We can see the similar investigations in [2], [4] and [10]. In this paper, we continue to study some properties of local cohomology modules which are belong to a Serre subcategory $\mathcal{S}$ of the category of R-modules. Since the following Serre subcategories of modules satisfy the condition $\mathcal{C}_{\mathfrak{a}}$ for all ideals $\mathfrak{a}$ of R, our results are valid in these classes:

i) $\mathcal{S}_0 :=$ the class of zero modules,

ii) $\mathcal{S}_A :=$ the class of Artinian modules,

iii) $\mathcal{S}_F :=$ the class of R-modules with finite support,

iv) $\mathcal{S}_n :=$ the class of R-modules M with $\dim_R M \leq n$, where n is a non-negative integer.

In one of our main results, we show that if $\mathcal{S}$ is a Serre subcategory of the category of R-modules such that $\mathcal{S} \in \mathcal{C}_{\mathfrak{a}_M}$ where $\mathfrak{a}_M := \operatorname{Ann}_R(M/\mathfrak{a}M)$, then for a finitely generated R-module M and an arbitrary R-module N we have

$$\inf\{i : \mathrm{H}^i_{\mathfrak{a}}(M,N) \text{ is not in } \mathcal{S}\} = \inf\{i : \mathrm{H}^i_{\mathfrak{a}_M}(N) \text{ is not in } \mathcal{S}\}.$$

Note that this result has been proved under certain conditions for the class of zero modules in [6], for the class of Artinian modules in [8] and for the class of R-modules with finite support in [9]. Here N is not necessarily finitely generated. As the dual case of this result, we prove that if M and N are finitely generated R-modules with $\mathrm{pd}(M) < \infty$ and $\mathcal{S} \in \mathcal{C}_{\mathfrak{a}_N}$ and also $\operatorname{Ann}_R N$ is in $\mathcal{S}$, then

$$\sup\{i : \mathrm{H}^i_{\mathfrak{a}}(M,N) \text{ is not in } \mathcal{S}\} = \sup\{i : \mathrm{H}^i_{\mathfrak{a}_N}(M,R) \text{ is not in } \mathcal{S}\}.$$

1 THE RESULTS

In this section, $\mathcal{S}$ is a Serre subcategory of the category of R-modules.

Definition 1.1. (See [1, Definition 2.1]) Let $\mathfrak{a}$ be an ideal of R. We say that $\mathcal{S}$ satisfies the condition $\mathcal{C}_{\mathfrak{a}}$ if for every $\mathfrak{a}$-torsion R-module M such that $0 :_M \mathfrak{a}$ is in $\mathcal{S}$, then M is in $\mathcal{S}$. In this case we write $\mathcal{S} \in \mathcal{C}_{\mathfrak{a}}$.

Lemma 1.2. Let M be a finitely generated R-module and N be an R-module. If N is in $\mathcal{S}$, then $\operatorname{Ext}^i_R(M,N)$ is in $\mathcal{S}$ for all $i \geq 0$.

Proof. See [4, Lemma 2.1]. $\boxed{QED}$

Lemma 1.3. Let $\mathfrak{a}$ be an ideal of R and let N be an $\mathfrak{a}$-torsion R-module. Then for every R-module M, $\mathrm{H}^i_{\mathfrak{a}}(M,N) \simeq \operatorname{Ext}^i_R(M,N)$ for all $i \geq 0$.

Proof. See [3, Lemma 3]. $\boxed{QED}$

Lemma 1.4. Let M be a finitely generated R-module, N be an R-module and $\mathfrak{a}$ be an ideal of R. Then

i) $\operatorname{Hom}_R(M, \mathrm{H}^0_{\mathfrak{a}}(N)) \simeq \mathrm{H}^0_{\mathfrak{a}}(\operatorname{Hom}_R(M,N)) \simeq \mathrm{H}^0_{\mathfrak{a}}(M,N)$.

ii) $\mathrm{H}^i_{\mathfrak{a}}(M,N) \simeq \mathrm{H}^i_{\mathfrak{a}_M}(M,N)$ for all $i \geq 0$. If N is finitely generated, then $\mathrm{H}^i_{\mathfrak{a}}(M,N) \simeq \mathrm{H}^i_{\mathfrak{a}_N}(M,N)$ for all $i \geq 0$.

Proof. i)
$$\begin{aligned}\operatorname{Hom}_R(M, \mathrm{H}^0_{\mathfrak{a}}(N)) &\simeq \operatorname{Hom}_R(M, \varinjlim_n \operatorname{Hom}_R(R/\mathfrak{a}^n, N))\\ &\simeq \varinjlim_n \operatorname{Hom}_R(M, \operatorname{Hom}_R(R/\mathfrak{a}^n, N))\\ &\simeq \varinjlim_n \operatorname{Hom}_R(R/\mathfrak{a}^n, \operatorname{Hom}(M, N))\\ &\simeq \mathrm{H}^0_{\mathfrak{a}}(\operatorname{Hom}_R(M, N))\\ &\simeq \varinjlim_n \operatorname{Hom}_R(R/\mathfrak{a}^n, \operatorname{Hom}(M, N))\\ &\simeq \varinjlim_n \operatorname{Hom}_R(M/\mathfrak{a}^n M, N)\\ &\simeq \mathrm{H}^0_{\mathfrak{a}}(M, N).\end{aligned}$$

ii) By [11, 9.23] $\sqrt{\operatorname{Ann}_R(M/\mathfrak{a}M)} = \sqrt{\mathfrak{a} + \operatorname{Ann}_R(M)}$ and so $\mathrm{H}^i_{\mathfrak{a}_M}(M, N) \simeq \mathrm{H}^i_{\mathfrak{a}+\operatorname{Ann}_R(M)}(M, N)$. But $\mathrm{H}^i_{\mathfrak{a}+\operatorname{Ann}_R(M)}(M, N) \simeq \mathrm{H}^i_{\mathfrak{a}}(M, N)$. Thus $\mathrm{H}^i_{\mathfrak{a}}(M, N) \simeq \mathrm{H}^i_{\mathfrak{a}_M}(M, N)$. Similarly we obtain the second result. $\boxed{QED}$

Theorem 1.5. Let L be a finitely generated R-module and N be an arbitrary R-module and let $n > 0$ be an integer. Then, if $\mathrm{H}^i_{\mathfrak{a}}(L, N)$ is in $\mathcal{S}$ for all $i < n$, then $\mathrm{H}^i_{\mathfrak{a}}(M, N)$ is in $\mathcal{S}$ for all $i < n$ and for all finitely generated R-modules M such that $\operatorname{Supp}_R(M) \subseteq \operatorname{Supp}_R(L)$.

Proof. We use induction on n. By [5, Lemma 2.2], there is a chain

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_t = M$$

of submodules of M such that the factors M_j/M_{j-1} are homomorphic image of a direct sum of finitely many copies of L. Take $j \in \{1, \cdots, t\}$ and set $V_j := M_j/M_{j-1}$. We have an exact sequence $0 \to U \to \oplus_{i=1}^l L \to V_j \to 0$, where l is an integer and U is a finitely generated R-module. Thus we have the following long exact sequence

$$0 \to \mathrm{H}^0_{\mathfrak{a}}(V_j, N) \to \mathrm{H}^0_{\mathfrak{a}}(\oplus_{i=1}^l L, N) \to \cdots \to \mathrm{H}^i_{\mathfrak{a}}(V_j, N) \to \mathrm{H}^i_{\mathfrak{a}}(\oplus_{i=1}^l L, N) \to \cdots.$$

Since $\mathrm{H}^0_{\mathfrak{a}}(\oplus_{i=1}^l L, N) \simeq \oplus_{i=1}^l \mathrm{H}^0_{\mathfrak{a}}(L, N)$ is in $\mathcal{S}$, from the above long exact sequence we conclude that $\mathrm{H}^0_{\mathfrak{a}}(V_j, N)$ is in $\mathcal{S}$ for all $j \in \{1, \cdots, t\}$. Now, from the exact sequence

$$0 \to M_1 \to M_2 \to V_2 \to 0$$

we have the following long exact sequence

$$0 \to \mathrm{H}^0_{\mathfrak{a}}(V_2, N) \to \mathrm{H}^0_{\mathfrak{a}}(M_2, N) \to \mathrm{H}^0_{\mathfrak{a}}(M_1, N) \to \cdots.$$

Since $M_1 = V_1$ by the above argument we have that $\mathrm{H}^0_{\mathfrak{a}}(M_1, N)$ is in $\mathcal{S}$ and $\mathrm{H}^0_{\mathfrak{a}}(V_2, N)$ is in $\mathcal{S}$. Thus by the above long exact sequence it follows that $\mathrm{H}^0_{\mathfrak{a}}(M_2, N)$ is in $\mathcal{S}$. By repeating this method we conclude that $\mathrm{H}^0_{\mathfrak{a}}(M_t, N)$ is in $\mathcal{S}$. Since

$M = M_t$ we conclude that $\mathrm{H}^0_{\mathfrak{a}}(M, N)$ is in $\mathcal{S}$. Hence the assertion holds for $n = 1$.

Now, let $n > 1$. Since $\mathrm{Supp}_R U \subseteq \mathrm{Supp}_R L$, by the induction hypothesis, $\mathrm{H}^i_{\mathfrak{a}}(U, N)$ is in $\mathcal{S}$ for all $i < n - 1$. On the other hand, $\mathrm{H}^i_{\mathfrak{a}}(\oplus_{i=1}^l L, N) \simeq \oplus_{i=1}^l \mathrm{H}^i_{\mathfrak{a}}(L, N)$ is in $\mathcal{S}$, for all $i < n$. Therefore, the first long exact sequence in this proof shows that $\mathrm{H}^i_{\mathfrak{a}}(V_j, N)$ is in $\mathcal{S}$ for all $i < n$. Thus we showed that the result holds for all R-modules $V_j := M_j/M_{j-1}$ where $1 \leq j \leq t$. Now, from the exact sequence

$$0 \to M_1 \to M_2 \to V_2 \to 0$$

we have the following long exact sequence

$$\cdots \to \mathrm{H}^i_{\mathfrak{a}}(V_2, N) \to \mathrm{H}^i_{\mathfrak{a}}(M_2, N) \to \mathrm{H}^i_{\mathfrak{a}}(M_1, N) \to \cdots .$$

Since $M_1 = V_1$ by the above long exact sequence it follows that $\mathrm{H}^i_{\mathfrak{a}}(M_2, N)$ is in $\mathcal{S}$ for all $i < n$. By repeating this method we conclude that $\mathrm{H}^i_{\mathfrak{a}}(M_t, N)$ is in $\mathcal{S}$ for all $i < n$. But $M = M_t$ and so the proof is complete. $\boxed{QED}$

The following corollaries are immediate cosequences of the above theorem.

Corollary 1.6. Let L and M be finitely generated R-modules and N be an R-module such that $\mathrm{Supp}_R(M) = \mathrm{Supp}_R(L)$. Let $n > 0$ be an integer. Then $\mathrm{H}^i_{\mathfrak{a}}(L, N)$ is in $\mathcal{S}$ for all $i < n$ if and only if $\mathrm{H}^i_{\mathfrak{a}}(M, N)$ is in $\mathcal{S}$ for all $i < n$.

Corollary 1.7. Let M be a finitely generated R-module and N be an R-module. Let $n > 0$ be an integer. If $\mathrm{H}^i_{\mathfrak{a}}(N)$ is in $\mathcal{S}$ for all $i < n$, then $\mathrm{H}^i_{\mathfrak{a}}(M, N)$ is in $\mathcal{S}$ for all $i < n$.

Proof. Since $\mathrm{H}^i_{\mathfrak{a}}(N) = \mathrm{H}^i_{\mathfrak{a}}(R, N)$ and $\mathrm{Supp}\, M \subseteq \mathrm{Supp}\, R$, the result follows by Theorem 1.5. $\boxed{QED}$

Now, we can prove the following main result.

Theorem 1.8. Let M be a finitely generated R-module, N be an R-module and $\mathfrak{a}$ be an ideal of R. Let $\mathcal{S} \in \mathcal{C}_{\mathfrak{a}_M}$ and let $n > 0$ be an integer. Then $\mathrm{H}^i_{\mathfrak{a}}(M, N)$ is in $\mathcal{S}$ for all $i < n$ if and only if $\mathrm{H}^i_{\mathfrak{a}_M}(N)$ is in $\mathcal{S}$ for all $i < n$.

Proof. We use induction on n. Let $n = 1$. If $\mathrm{H}^0_{\mathfrak{a}_M}(N)$ is in $\mathcal{S}$, then we conclude that $\mathrm{Hom}_R(M, \mathrm{H}^0_{\mathfrak{a}_M}(N))$ is in $\mathcal{S}$ by Lemma 1.2 and so $\mathrm{H}^0_{\mathfrak{a}}(M, N)$ is in $\mathcal{S}$ by Lemma 1.4.

Now, assume that $\mathrm{H}^0_{\mathfrak{a}}(M, N)$ is in $\mathcal{S}$. By Lemma 1.4, $\mathrm{H}^0_{\mathfrak{a}_M}(\mathrm{Hom}_R(M, N))$ is in $\mathcal{S}$. Since

$$\mathrm{H}^0_{\mathfrak{a}_M}(\mathrm{Hom}_R(M, N)) = \cup_{t>0} \mathrm{Hom}_R(R/(\mathfrak{a}_M)^t, \mathrm{Hom}_R(M, N))$$

we conclude that $\mathrm{Hom}_R(R/\mathfrak{a}_M, \mathrm{Hom}_R(M,N))$ is in $\mathcal{S}$ and so $\mathrm{Hom}_R(M/\mathfrak{a}_M M, N)$ is in $\mathcal{S}$. Again Lemma 1.4 implies that $\mathrm{H}^0_{\mathfrak{a}_M}(M/\mathfrak{a}_M M, N)$ is in $\mathcal{S}$.

Since $\mathrm{Supp}_R(M/\mathfrak{a}_M M) = \mathrm{Supp}_R(R/\mathfrak{a}_M)$, we have

$$\mathrm{H}^0_{\mathfrak{a}_M}(R/\mathfrak{a}_M, N) \simeq \mathrm{Hom}_R(R/\mathfrak{a}_M, \mathrm{H}^0_{\mathfrak{a}_M}(N))$$

is in $\mathcal{S}$ by Corollary 1.6. But by hypothesis $\mathcal{S} \in \mathcal{C}_{\mathfrak{a}_M}$ and so $\mathrm{H}^0_{\mathfrak{a}_M}(N)$ is in $\mathcal{S}$.

Now, suppose that $n > 1$. The exact sequence

$$0 \to \Gamma_{\mathfrak{a}_M}(N) \to N \to N/\Gamma_{\mathfrak{a}_M}(N) \to 0$$

induces the exact sequence

$$\mathrm{H}^i_{\mathfrak{a}}(M, \Gamma_{\mathfrak{a}_M}(N)) \to \mathrm{H}^i_{\mathfrak{a}}(M,N) \to \mathrm{H}^i_{\mathfrak{a}}(M, N/\Gamma_{\mathfrak{a}_M}(N)) \to \mathrm{H}^{i+1}_{\mathfrak{a}}(M, \Gamma_{\mathfrak{a}_M}(N)) \to \cdots .$$

Since $\mathfrak{a} \subseteq \mathfrak{a}_M$, $\Gamma_{\mathfrak{a}_M}(N)$ is an $\mathfrak{a}$-torsion R-module and so by Lemma 1.3 we get the following exact sequence

$$\mathrm{Ext}^i_R(M, \Gamma_{\mathfrak{a}_M}(N)) \to \mathrm{H}^i_{\mathfrak{a}}(M,N) \to \mathrm{H}^i_{\mathfrak{a}}(M, N/\Gamma_{\mathfrak{a}_M}(N)) \to \mathrm{Ext}^{i+1}_R(M, \Gamma_{\mathfrak{a}_M}(N)) \to .$$

The two outer terms belong to $\mathcal{S}$ by Lemma 1.2 and thus $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ is in $\mathcal{S}$ if and only if $\mathrm{H}^i_{\mathfrak{a}}(M, N/\Gamma_{\mathfrak{a}_M}(N))$ is in $\mathcal{S}$ for all $i \geq 0$. On the other hand, the above short exact sequence induces the exact sequence

$$0 \to \mathrm{H}^0_{\mathfrak{a}_M}(\Gamma_{\mathfrak{a}_M}(N)) \to \mathrm{H}^0_{\mathfrak{a}_M}(N) \to \mathrm{H}^0_{\mathfrak{a}_M}(N/\Gamma_{\mathfrak{a}_M}(N)) \to \mathrm{H}^1_{\mathfrak{a}_M}(\Gamma_{\mathfrak{a}_M}(N)) \to \cdots .$$

Since $\mathrm{H}^0_{\mathfrak{a}_M}(\Gamma_{\mathfrak{a}_M}(N))$ is a submodule of $\Gamma_{\mathfrak{a}_M}(N)$, it follows that $\mathrm{H}^0_{\mathfrak{a}_M}(\Gamma_{\mathfrak{a}_M}(N))$ is in $\mathcal{S}$. But by [7, Corollary 2.1.7 (ii)] $\mathrm{H}^1_{\mathfrak{a}_M}(\Gamma_{\mathfrak{a}_M}(N)) = 0$. Thus the above sequence shows that $\mathrm{H}^0_{\mathfrak{a}_M}(N)$ is in $\mathcal{S}$ if and only if $\mathrm{H}^0_{\mathfrak{a}_M}(N/\Gamma_{\mathfrak{a}_M}(N))$ is in $\mathcal{S}$. But $\mathrm{H}^i_{\mathfrak{a}_M}(N) \simeq \mathrm{H}^i_{\mathfrak{a}_M}(N/\Gamma_{\mathfrak{a}_M}(N))$ for all $i > 0$. Therefore, $\mathrm{H}^i_{\mathfrak{a}_M}(N)$ is in $\mathcal{S}$ if and only if $\mathrm{H}^i_{\mathfrak{a}_M}(N/\Gamma_{\mathfrak{a}_M}(N))$ is in $\mathcal{S}$ for all $i \geq 0$. So we can assume that $\mathrm{H}^0_{\mathfrak{a}_M}(N) = 0$. Let E be the injective hull of N. Then $\mathrm{H}^0_{\mathfrak{a}_M}(E) = 0$ and $\mathrm{H}^0_{\mathfrak{a}_M}(M,E) \simeq \mathrm{Hom}_R(M, \mathrm{H}^0_{\mathfrak{a}_M}(E)) = 0$. Hence from the exact sequence

$$0 \to N \to E \to E/N \to 0$$

we get $\mathrm{H}^{i+1}_{\mathfrak{a}_M}(N) \simeq \mathrm{H}^i_{\mathfrak{a}_M}(E/N)$ and $\mathrm{H}^{i+1}_{\mathfrak{a}_M}(M,N) \simeq \mathrm{H}^i_{\mathfrak{a}_M}(M, E/N)$ for all $i \geq 0$. Since $\mathrm{H}^i_{\mathfrak{a}_M}(M,N) \simeq \mathrm{H}^i_{\mathfrak{a}}(M,N)$ for all $i \geq 0$, we have $\mathrm{H}^{i+1}_{\mathfrak{a}}(M,N) \simeq \mathrm{H}^i_{\mathfrak{a}}(M, E/N)$ for all $i \geq 0$. Now, the induction completes the proof. $\boxed{QED}$

Corollary 1.9. Let M be a finitely generated R-module and N be an R-module and $\mathfrak{a}$ an ideal of R. Let $\mathcal{S} \in \mathcal{C}_{\mathfrak{a}_M}$. Then

$$\inf\{i : \mathrm{H}^i_{\mathfrak{a}}(M,N) \text{ is not in } \mathcal{S}\} = \inf\{i : \mathrm{H}^i_{\mathfrak{a}_M}(N) \text{ is not in } \mathcal{S}\}.$$

The following results (i), (ii) and (iii) are generalizations of [6, Proposition 5.5], [8, Theorem 2.2] and [9, Corollary 2.9] for every R-module N (not necessarily finitely generated).

Corollary 1.10. Let M be a finitely generated R-module and N be an R-module. Then we have:

i) $\inf\{i : \mathrm{H}^i_{\mathfrak{a}}(M,N) \neq 0\} = \inf\{i : \mathrm{H}^i_{\mathfrak{a}_M}(N) \neq 0\}$.

ii) $\inf\{i : \mathrm{H}^i_{\mathfrak{a}}(M,N)$ is not Artinian $\} = \inf\{i : \mathrm{H}^i_{\mathfrak{a}_M}(N)$ is not Artinian$\}$.

iii) $\inf\{i : \mathrm{Supp}_R(\mathrm{H}^i_{\mathfrak{a}}(M,N))$ is not finite $\} = \inf\{i : \mathrm{Supp}_R(\mathrm{H}^i_{\mathfrak{a}_M}(N))$ is not finite $\}$.

iv) If n is a non-negative integer, then we have

$$\inf\{i : \dim_R \mathrm{H}^i_{\mathfrak{a}}(M,N) > n\} = \inf\{i : \dim_R(\mathrm{H}^i_{\mathfrak{a}_M}(N)) > n\}.$$

Proof. The result follows by using Corollary 1.9 for the classes of $\mathcal{S}_0$, $\mathcal{S}_A$, $\mathcal{S}_F$ and $\mathcal{S}_n$. $\boxed{QED}$

Theorem 1.11. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R such that $\mathfrak{a} \subseteq \mathfrak{b}$ and let $\mathcal{S} \in \mathcal{C}_{\mathfrak{b}}$. Let N be an R-module and let $n > 0$ be an integer. If $\mathrm{H}^i_{\mathfrak{a}}(N)$ is in $\mathcal{S}$ for all $i < n$, then $\mathrm{H}^i_{\mathfrak{b}}(N)$ is in $\mathcal{S}$ for all $i < n$.

Proof. By Corollary 1.7, $\mathrm{H}^i_{\mathfrak{a}}(R/\mathfrak{b}, N)$ is in $\mathcal{S}$ for all $i < n$. Since $\mathfrak{a} + \mathrm{Ann}(R/\mathfrak{b}) = \mathfrak{a} + \mathfrak{b}$, Theorem 1.8 implies that $\mathrm{H}^i_{\mathfrak{a}+\mathfrak{b}}(N)$ is in $\mathcal{S}$ for all $i < n$. But $\mathfrak{a} \subseteq \mathfrak{b}$ and so $\mathrm{H}^i_{\mathfrak{b}}(N)$ is in $\mathcal{S}$ for all $i < n$, as required. $\boxed{QED}$

Corollary 1.12. Let M be a finitely generated R-module and N be an R-module and let $\mathfrak{a}$ be an ideal of R. Let $\mathcal{S} \in \mathcal{C}_{\mathfrak{a}}$. If N is in $\mathcal{S}$, then $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ is in $\mathcal{S}$ for all $i \geq 0$.

Proof. Since $\mathrm{H}^i_0(N) = N$ for $i = 0$ and $\mathrm{H}^i_0(N) = 0$ for all $i > 0$ and N is in $\mathcal{S}$, we conclude that $\mathrm{H}^i_0(N)$ is in $\mathcal{S}$ for all $i \geq 0$ and so Theorem 1.11 implies that $\mathrm{H}^i_{\mathfrak{a}}(N)$ is in $\mathcal{S}$ for all $i \geq 0$. Now, the result follows by Corollary 1.7. $\boxed{QED}$

Corollary 1.13. Let M be an R-module and let $\mathfrak{a}$ be a non-zero ideal of R and let $\mathcal{S} \in \mathcal{C}_{\mathfrak{a}}$. Let $x \in \mathfrak{a}$ be an arbitrary element. If $\mathrm{H}^0_{\langle x \rangle}(M)$ and $\mathrm{H}^1_{\langle x \rangle}(M)$ are in $\mathcal{S}$, then $\mathrm{H}^i_{\mathfrak{a}}(M)$ is in $\mathcal{S}$ for all $i \geq 0$.

Proof. By [7, Theorem 3.3.1], $\mathrm{H}^i_{\langle x \rangle}(M) = 0$ for all $i > 1$. Thus $\mathrm{H}^i_{\langle x \rangle}(M)$ is in $\mathcal{S}$ for all $i \geq 0$. Since $\langle x \rangle \subseteq \mathfrak{a}$, we have $\mathrm{H}^i_{\mathfrak{a}}(M)$ is in $\mathcal{S}$ for all $i \geq 0$ by Theorem 1.11. $\boxed{QED}$

Corollary 1.14. Let M be an R-module and $\mathfrak{a}$ be a non-zero ideal of R. Then $\mathrm{Supp}_R \mathrm{H}^i_{\mathfrak{a}}(M) \subseteq \cap_{x \in \mathfrak{a}}(\mathrm{Supp}_R \mathrm{H}^0_{\langle x \rangle}(M) \cup \mathrm{Supp}_R \mathrm{H}^1_{\langle x \rangle}(M))$ for all $i \geq 0$.

Proof. Let $\mathfrak{p} \in \operatorname{Supp}_R \mathrm{H}^i_{\mathfrak{a}}(M)$. Then $\mathrm{H}^i_{\mathfrak{a}_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0$. Using the above corollary for the class of zero modules, we have $\mathrm{H}^0_{\langle x\rangle R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0$ or $\mathrm{H}^1_{\langle x\rangle R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0$ for all $x \in \mathfrak{a}$. Thus $\mathfrak{p} \in \operatorname{Supp}_R \mathrm{H}^0_{\langle x\rangle}(M)$ or $\mathfrak{p} \in \operatorname{Supp}_R \mathrm{H}^1_{\langle x\rangle}(M)$ for all $x \in \mathfrak{a}$. $\boxed{QED}$

By using the Independence Theorem, [7, 4.2.1], we get the next result.

Theorem 1.15. Let M be a finitely generated R-module and N be an R-module. Let $\mathfrak{a}$ be an ideal of R and let $n > 0$ be an integer. If $\operatorname{Ann}_R(M) \subseteq \operatorname{Ann}_R(N)$ and $R/\operatorname{Ann}_R(M)$ is a flat R-module, then the following are equivalent:

i) $\mathrm{H}^i_{\mathfrak{a}}(N)$ is in $\mathcal{S}$ for all $i < n$.

ii) $\mathrm{H}^i_{\mathfrak{a}}(M, N)$ is in $\mathcal{S}$ for all $i < n$.

Proof. i) $\Rightarrow$ ii): By Corollary 1.7.

ii) $\Rightarrow$ i): Let $\mathrm{H}^i_{\mathfrak{a}}(M, N)$ be in $\mathcal{S}$ for all $i < n$. Then by Corollary 1.6, $\mathrm{H}^i_{\mathfrak{a}}(R/\operatorname{Ann}_R(M), N)$ is in $\mathcal{S}$ for all $i < n$. Take $R' := R/\operatorname{Ann}_R(M)$. Since $\operatorname{Ann}_R(M) \subseteq \operatorname{Ann}_R(N)$, N is an R'-module. But $f : R \to R'$ is a flat ring homomorphism, and so by [6, Proposition 3.1] there exists an R-isomorphism $\mathrm{H}^i_{\mathfrak{a}R'}(R', N) \simeq \mathrm{H}^i_{\mathfrak{a}}(R', N)$ for all $i \geq 0$. But $\mathrm{H}^i_{\mathfrak{a}R'}(R', N) \simeq \mathrm{H}^i_{\mathfrak{a}R'}(N)$ as R'-modules. By the Independence Theorem ([7, Theorem 4.2.1]), there exists an R-isomorphism $\mathrm{H}^i_{\mathfrak{a}R'}(N) \simeq \mathrm{H}^i_{\mathfrak{a}}(N)$ for all $i \geq 0$. Thus $\mathrm{H}^i_{\mathfrak{a}}(N) \simeq \mathrm{H}^i_{\mathfrak{a}}(R', N)$ as R-modules. Since $\mathrm{H}^i_{\mathfrak{a}}(R', N)$ is in $\mathcal{S}$ for all $i < n$, $\mathrm{H}^i_{\mathfrak{a}}(N)$ is in $\mathcal{S}$ for all $i < n$, as required. $\boxed{QED}$

Corollary 1.16. Let M be a finitely generated R-module and N be an R-module. Let $\mathfrak{a}$ be an ideal of R and $n > 0$ be an integer. If $\operatorname{Ann}_R(M) \subseteq \operatorname{Ann}_R(N)$ and $R/\operatorname{Ann}_R(M)$ is a flat R-module, then $\cup_{i<n} \operatorname{Supp}_R \mathrm{H}^i_{\mathfrak{a}}(N) = \cup_{i<n} \operatorname{Supp}_R \mathrm{H}^i_{\mathfrak{a}}(M, N)$.

Proof. Let $\mathfrak{p} \notin \cup_{i<n} \operatorname{Supp}_R \mathrm{H}^i_{\mathfrak{a}}(N)$. Then $\mathrm{H}^i_{\mathfrak{a}R_p}(N_{\mathfrak{p}}) = 0$ for all $i < n$. Since $\operatorname{Ann}_R(M) \subseteq \operatorname{Ann}_R(N)$ we have $\operatorname{Ann}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \subseteq \operatorname{Ann}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}})$. Thus Theorem 1.15, for the class of zero modules, implies that $\mathrm{H}^i_{\mathfrak{a}R_p}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) = 0$ for all $i < n$ and it follows that $\mathfrak{p} \notin \cup_{i<n} \operatorname{Supp}_R \mathrm{H}^i_{\mathfrak{a}}(M, N)$. Therefore, $\cup_{i<n} \operatorname{Supp}_R \mathrm{H}^i_{\mathfrak{a}}(M, N) \subseteq \cup_{i<n} \operatorname{Supp}_R \mathrm{H}^i_{\mathfrak{a}}(N)$. Similarly, we can prove the converse inclusion. $\boxed{QED}$

In the remainder of this paper, we investigate the local cohomology modules $\mathrm{H}^i_{\mathfrak{a}}(M, N)$ for all $i > n$ where M and N are finitely generated R-modules and n is an integer.

Theorem 1.17. Let L and M be finitely generated R-modules with $\operatorname{pd}(M) < \infty$. Let $n \geq 0$ be an integer. Then, if $\mathrm{H}^i_{\mathfrak{a}}(M, L)$ is in $\mathcal{S}$ for all $i > n$, then $\mathrm{H}^i_{\mathfrak{a}}(M, N)$ is in $\mathcal{S}$ for all $i > n$ and for all finitely generated R-modules N such that $\operatorname{Supp}_R N \subseteq \operatorname{Supp}_R L$.

Proof. Let $t := \operatorname{pd} M + ara(\mathfrak{a})$, where $ara(\mathfrak{a})$ the arithmetic rank of the ideal $\mathfrak{a}$ is the least number of elements of R required to generate an ideal which has the same radical as $\mathfrak{a}$. For any finitely generated R-module C, we have $\mathrm{H}^i_{\mathfrak{a}}(M, C) = 0$ for all $i > t$, see [12, Theorem 2.5]. Thus we can assume that $n < i \leq t$. We use decreasing induction on i. By an argument similar to that used in the proof of Theorem 1.5, we obtain an exeact sequence $0 \to U \to \oplus_{i=1}^{l} L \to N \to 0$ for some $l > 0$ and some finitely generated R-module U. Thus, we get the following long exact sequence:

$$\cdots \to \mathrm{H}^i_{\mathfrak{a}}(M, \oplus_{i=1}^{l} L) \to \mathrm{H}^i_{\mathfrak{a}}(M, N) \to \mathrm{H}^{i+1}_{\mathfrak{a}}(M, U) \to \cdots .$$

Since $\operatorname{Supp}_R U \subseteq \operatorname{Supp}_R L$, we get by induction that $\mathrm{H}^{i+1}_{\mathfrak{a}}(M, U)$ is in $\mathcal{S}$. But, by the hypothesis, $\mathrm{H}^i_{\mathfrak{a}}(M, \oplus_{i=1}^{l} L) \simeq \oplus_{i=1}^{l} \mathrm{H}^i_{\mathfrak{a}}(M, L)$ is in $\mathcal{S}$. Therefore, the above long exact sequence implies that $\mathrm{H}^i_{\mathfrak{a}}(M, N)$ is in $\mathcal{S}$, as required. QED

Corollary 1.18. Let L, M and N be finitely generated R-modules with $\operatorname{pd}(M) < \infty$ and $\operatorname{Supp}_R N = \operatorname{Supp}_R L$. Let $\mathfrak{a}$ be an ideal of R and $n \geq 0$ be an integer. Then $\mathrm{H}^i_{\mathfrak{a}}(M, L)$ is in $\mathcal{S}$ for all $i > n$ if and only if $\mathrm{H}^i_{\mathfrak{a}}(M, N)$ is in $\mathcal{S}$ for all $i > n$.

Corollary 1.19. Let M and N be finitely generated R-modules with $\operatorname{pd}(M) < \infty$. Let $\mathfrak{a}$ be an ideal of R and let $n \geq 0$ be an integer. If $\mathrm{H}^i_{\mathfrak{a}}(M, R)$ is in $\mathcal{S}$ for all $i > n$, then $\mathrm{H}^i_{\mathfrak{a}}(M, N)$ is in $\mathcal{S}$ for all $i > n$.

Theorem 1.20. Let M and N be finitely generated R-modules with $\operatorname{pd}(M) < \infty$ and let $\mathfrak{a}$ be an ideal of R. Let $\mathcal{S} \in \mathcal{C}_{\mathfrak{a}}$ and let $n \geq 0$ be an integer. If $\operatorname{Ann}_R N$ is in $\mathcal{S}$, then the following conditions are equivalent:

i) $\mathrm{H}^i_{\mathfrak{a}}(M, R)$ is in $\mathcal{S}$ for all $i > n$.

ii) $\mathrm{H}^i_{\mathfrak{a}}(M, N)$ is in $\mathcal{S}$ for all $i > n$.

Proof. i) $\Rightarrow$ ii): By Corollary 1.19.

ii) $\Rightarrow$ i): Since $\operatorname{Supp}_R N = \operatorname{Supp}_R R/\operatorname{Ann}_R N$, we have that $\mathrm{H}^i_{\mathfrak{a}}(M, R/\operatorname{Ann}_R N)$ is in $\mathcal{S}$ for all $i > n$ by Corollary 1.18. Now, the exact sequence

$$0 \to \operatorname{Ann}_R N \to R \to R/\operatorname{Ann}_R N \to 0$$

induces the long exact sequence

$$\cdots \to \mathrm{H}^i_{\mathfrak{a}}(M, \operatorname{Ann}_R N) \to \mathrm{H}^i_{\mathfrak{a}}(M, R) \to \mathrm{H}^i_{\mathfrak{a}}(M, R/\operatorname{Ann}_R N) \to \cdots .$$

Since $\operatorname{Ann}_R N$ is in $\mathcal{S}$, $\mathrm{H}^i_{\mathfrak{a}}(M, \operatorname{Ann}_R N)$ is in $\mathcal{S}$ for all i by Corollary 1.12. Therefore, from the above long exact sequance we conclude that $\mathrm{H}^i_{\mathfrak{a}}(M, R)$ is in $\mathcal{S}$ for all $i > n$. QED

Now, we can obtain the next result which is a dual of Theorem 1.8 .

Corollary 1.21. Let M and N be finitely generated R-modules with $\mathrm{pd}(M) < \infty$ and let $\mathfrak{a}$ be an ideal of R. Let $\mathcal{S} \in \mathcal{C}_{\mathfrak{a}_N}$ and let $n \geq 0$ be an integer. If $\mathrm{Ann}_R N$ is in $\mathcal{S}$, then the following conditions are equivalent:

i) $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ is in $\mathcal{S}$ for all $i > n$.

ii) $\mathrm{H}^i_{\mathfrak{a}+\mathrm{Ann}_R N}(M,R)$ is in $\mathcal{S}$ for all $i > n$.

Therefore,

$$\sup\{i : \mathrm{H}^i_{\mathfrak{a}}(M,N) \text{ is not in } \mathcal{S}\} = \sup\{i : \mathrm{H}^i_{\mathfrak{a}+\mathrm{Ann}_R N}(M,R) \text{ is not in } \mathcal{S}\}.$$

Proof. By Lemma 1.4 $\mathrm{H}^i_{\mathfrak{a}}(M,N) \simeq \mathrm{H}^i_{\mathfrak{a}+\mathrm{Ann}\,N}(M,N) \simeq \mathrm{H}^i_{\mathfrak{a}_N}(M,N)$ for all $i \geq 0$, and so the result follows by Theorem 1.20. $\boxed{QED}$

Corollary 1.22. Let M be a finitely generated R-module with $\mathrm{pd}(M) < \infty$ and let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R such that $\mathfrak{a} \subseteq \mathfrak{b}$. Let $\mathcal{S} \in \mathcal{C}_{\mathfrak{b}}$ and $\mathfrak{b}$ be in $\mathcal{S}$. Let $n \geq 0$ be an integer. If $\mathrm{H}^i_{\mathfrak{a}}(M,R)$ is in $\mathcal{S}$ for all $i > n$, then $\mathrm{H}^i_{\mathfrak{b}}(M,R)$ is in $\mathcal{S}$ for all $i > n$.

Proof. By Theorem 1.20, $\mathrm{H}^i_{\mathfrak{a}}(M,R/\mathfrak{b})$ is in $\mathcal{S}$ for all $i > n$ and by Corollary 1.21, $\mathrm{H}^i_{\mathfrak{a}+\mathfrak{b}}(M,R)$ is in $\mathcal{S}$ for all $i > n$. Since $\mathfrak{a} \subseteq \mathfrak{b}$ we get $\mathrm{H}^i_{\mathfrak{b}}(M,R)$ is in $\mathcal{S}$ for all $i > n$, as required. $\boxed{QED}$

Corollary 1.23. Let M be a finitely generated R-module with $\mathrm{pd}(M) < \infty$ and let $\mathfrak{a}$ be an ideal of R. Let $\mathcal{S} \in \mathcal{C}_{\mathfrak{a}}$ and $\mathfrak{a}$ be in $\mathcal{S}$. Let $n \geq 0$ be an integer. Then the following conditions are equivalent:

i) $\mathrm{H}^i_{\mathfrak{a}}(M,R)$ is in $\mathcal{S}$ for all $i > n$.

ii) $\mathrm{Ext}^i_R(M,R/\mathfrak{a})$ is in $\mathcal{S}$ for all $i > n$.

iii) $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ is in $\mathcal{S}$ for each finitely generated R-module N and for all $i > n$.

Proof. i)$\Rightarrow$ ii): By hypothesis $\mathrm{H}^i_{0+\mathfrak{a}}(M,R)$ is in $\mathcal{S}$ for all $i > n$. Thus by Corollary 1.21, $\mathrm{H}^i_0(M,R/\mathfrak{a})$ is in $\mathcal{S}$ for all $i > n$. Since $\mathrm{H}^i_0(M,R/\mathfrak{a}) \simeq \mathrm{Ext}^i_R(M,R/\mathfrak{a})$, we have $\mathrm{Ext}^i_R(M,R/\mathfrak{a})$ is in $\mathcal{S}$ for all $i > n$.

ii) $\Rightarrow$ i): If $\mathrm{Ext}^i_R(M,R/\mathfrak{a})$ is in $\mathcal{S}$ for all $i > n$, then $\mathrm{H}^i_0(M,R/\mathfrak{a})$ is in $\mathcal{S}$ for all $i > n$ and by Corollary 1.21, $\mathrm{H}^i_{0+\mathfrak{a}}(M,R)$ is in $\mathcal{S}$ for all $i > n$, as desired.

i) $\Rightarrow$ iii): By Theorem 1.20.

iii) $\Rightarrow$ i): It is clear. $\boxed{QED}$

Acknowledgements. The author would like to thank the referees for their careful reading and helpful suggestions on this paper.

References

[1] M. Aghapournahr, L. Melkersson: Local cohomology and Serre subcategories, J. Algebra. 320 (2008), 1275-1287.

[2] M. Aghapournahr, L. Melkersson: A natural map in local cohomology, Ark. Mat. 48(2) (2010), 243-251.

[3] J. Amjadi and R. Naghipour: Cohomological dimension of generalized local cohomology modules, Algebra Colloq., 15(2) (2008), 303- 308.

[4] M. Asgharzadeh and M. Tousi: A unified approach to local cohomology modules using serre classes, Canad. Math. Bull., 53 (2010), 577- 586.

[5] K. Bahmanpour: Cohomological dimension, cofiniteness and Abelian categories of cofinte modules, J. Algebra, 484 (2017), 168-197.

[6] M. H. Bijan-Zadeh: A common generalization of local cohomology theories, Glascow Math. J., 21(2) (1980), 173-181.

[7] M. Brodmann and R.Y. Sharp: Local cohomology: an algebraic introduction with geometric applications, Cambridge Univ. Press, 60 (1998).

[8] L. Chu and Z. Tang: On the Artinianness of generalized local cohomology, Comm. Algebra, 35(12) (2007), 3821-3827.

[9] N. T. Coung and N. V. Hoang: On the vanishing and the finiteness of supports of generalized local cohomology modules, Manuscripta math, 126 (2008), 59-72.

[10] S. O. Faramarzi, Z. Barghsouz: Serre subcategory, local homology and local cohomology, J. Algebraic Systems, 7(2) (2020), 301-314.

[11] R. Y. Sharp: Steps in commutative algebra, London Mathematical Society student Texs 19, Cambridge University Press, 1990.

[12] S. Yassemi: Generalized section functors, J. Pure. Apple. Algebra, 95 (1994), 103-119.