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On certain new theta function identities of level-13 involving Ramanujan's functions $\varphi(-q)$ and $\psi(q)$

Kaliyur Ranganna Vasuki
Department of Studies in Mathematics, University of Mysore, Manasagangotri, Mysuru–570 006, INDIA.
vasuki_kr@hotmail.com

Nagendra Prabhu
Department of Mathematics, Sri. H. D. Devegowda Government First Grade College, Paduvalahippe, Hassan University, Hassan–573 211, INDIA
nagp149@gmail.com

Shibu Andia
Department of Mathematics, Vidyavardhaka College of Engineering, Mysuru–570 002, INDIA.
shibu.andia@gmail.com

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Abstract. On page 244 of his second notebook, Ramanujan recorded several remarkable identities involving his theta function $f(-q)$. Among these, one particularly elegant identity was noted by R. J. Evans to be the last proved in the second notebook. Motivated by these classical results, the present article investigates analogous identities associated with Ramanujan's other two well-known theta functions $\varphi(-q)$ and $\psi(q)$.

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1 Introduction

For any two complex numbers a and q, with $|q| < 1$, we define

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1 - aq^n).$$

On page 197 of his second notebook [7], Ramanujan defined his theta function

$f(a,b)$ as

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \quad |ab| < 1.$$

By the well known Jacobi's triple product identity, we have

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty,$$

The three well-known Ramanujan's theta functions are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty,$$
$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}} = (q;q)_\infty.$$

After Ramanujan, we define

$$\chi(q) := (-q;q^2)_\infty.$$

For convenience, we set $f_n = f(-q^n)$ and we call an identity involving the function $f(a,b)$ with $ab = q^k$ as a level-k theta function identity. On page 244 of his second notebook, Ramanujan recorded the following remarkable level-13 theta function identities involving one of his well-known theta function $f(-q)$:

$$G_1 + G_2 + G_3 + G_4 + G_5 + G_6 + 1 = \frac{f(-q^{\frac{1}{13}})}{q^{\frac{7}{13}} f(-q^{13})}, \tag{1.1}$$

$$G_1G_5 + G_2G_3 + G_4G_6 = -1 - \frac{f^2(-q)}{qf^2(-q^{13})}, \tag{1.2}$$

$$\frac{1}{G_1G_5} + \frac{1}{G_2G_3} + \frac{1}{G_4G_6} = 4 + \frac{f^2(-q)}{qf^2(-q^{13})}, \tag{1.3}$$

and

$$G_1G_3G_4 + G_2G_5G_6 = 3 + \frac{f^2(-q)}{qf^2(-q^{13})}, \tag{1.4}$$

where

$$G_i = (-1)^i q^{\frac{(i-2)(3i-7)-14}{26}} \frac{f(-q^{2i},-q^{13-2i})}{f(-q^i,-q^{13-i})}.$$

The identity (1.1) follows from one of the general formula [2, p. 274] and for proofs of the identities (1.2)–(1.4), one may refer to the works of R. J. Evans [4] and J. N. O'Brien [6]. It is interesting to note that the identity (1.3) is the last identity proved in Ramanujan's second notebook. Recently, a new and simple proofs of the identities (1.2)–(1.4) are given by K. R. Vasuki and N. Prabhu in [9].

Motivated by the above identities of Ramanujan, in the present article, We aim to derive new identities involving Ramanujan's theta functions $\varphi(-q)$ and $\psi(q)$ that are analogous to the identities (1.1)–(1.4), and thereby extending the spirit of Ramanujan's work. We establish them in the next section of this article.

2 Main results

We begin this section by recalling the following well-known Bailey's summation formula [1, Eq. 5]:

Lemma 2.1. We have

$$\sum_{n=-\infty}^{\infty}\frac{aq^n}{(1-aq^n)^2}-\sum_{n=-\infty}^{\infty}\frac{bq^n}{(1-bq^n)^2}=af^6(-q)\frac{f(-ab,-\frac{q}{ab})f(-\frac{b}{a},-\frac{aq}{b})}{f^2(-a,-\frac{q}{a})f^2(-b,-\frac{q}{b})}. \tag{2.1}$$

The following theta function identity is the well known Weierstrass' three-term identity, appears in the works of Tannery and Molk [8, p. 160], see also [3, p. 66].

Lemma 2.2. We have

$$\begin{aligned} &cf\left(-ab,-\frac{q}{ab}\right)f\left(-\frac{b}{a},-\frac{aq}{b}\right)f\left(-cd,-\frac{q}{cd}\right)f\left(-\frac{d}{c},-\frac{cq}{d}\right)\\ &-bf\left(-ac,-\frac{q}{ac}\right)f\left(-\frac{c}{a},-\frac{aq}{c}\right)f\left(-bd,-\frac{q}{bd}\right)f\left(-\frac{d}{b},-\frac{bq}{d}\right)\\ &+bf\left(-ad,-\frac{q}{ad}\right)f\left(-\frac{d}{a},-\frac{aq}{d}\right)f\left(-bc,-\frac{q}{bc}\right)f\left(-\frac{c}{b},-\frac{bq}{c}\right)=0. \end{aligned} \tag{2.2}$$

We now state and prove our main results of the present work.

Theorem 2.3. We have

$$1+2\left(E_1+E_2+E_3+E_4+E_5+E_6\right)=\frac{\varphi(-q^{\frac{1}{13}})}{\varphi(-q^{13})}, \tag{2.3}$$

$$E_1E_3+E_2E_6+E_4E_5 \quad =\frac{1}{4}\left(\frac{\varphi^2(-q)}{\varphi^2(-q^{13})}-1\right), \tag{2.4}$$

and

$$\begin{aligned}\frac{1}{E_1E_3}+\frac{1}{E_2E_6}+\frac{1}{E_4E_5}=&\frac{1}{2q^{\frac{14}{3}}}\frac{f_2^3f_{13}^5}{f_1f_{26}^7}\left(13q^2\frac{f_{26}^2}{f_2^2}+5+\frac{f_2^2}{q^2f_{26}^2}\right)^{\frac{2}{3}}\\&-\frac{1}{32q^7}\frac{f_1^3f_{13}^9}{f_2f_{26}^{11}}\left(\frac{\varphi^2(-q)}{\varphi^2(-q^{13})}-6+5\frac{\varphi^2(-q^{13})}{\varphi^2(-q)}\right),\end{aligned}\tag{2.5}$$

where

$$E_1=q^{\frac{36}{13}}\frac{f(-q,-q^{25})}{\varphi(-q^{13})}, E_2=-q^{\frac{25}{13}}\frac{f(-q^3,-q^{23})}{\varphi(-q^{13})}, E_3=q^{\frac{16}{13}}\frac{f(-q^5,-q^{21})}{\varphi(-q^{13})},$$
$$E_4=-q^{\frac{9}{13}}\frac{f(-q^7,-q^{19})}{\varphi(-q^{13})}, E_5=q^{\frac{4}{13}}\frac{f(-q^9,-q^{17})}{\varphi(-q^{13})},\ \textit{and}\ E_6=-q^{\frac{1}{13}}\frac{f(-q^{11},-q^{15})}{\varphi(-q^{13})}.$$

Proof. We have [2, p. 49]

$$\varphi(q)=\varphi(q^{n^2})+\sum_{r=1}^{n-1}q^{r^2}f(q^{n(n-2r)},q^{n(n+2r)}).$$

Setting $n=13$ in the above, then changing q to $-q^{\frac{1}{13}}$, and dividing the resulting equation by $\varphi(-q^{13})$, we obtain the required identity (2.3). By the definition of $\varphi(q)$, we have

$$\varphi^2(q)=\sum_{m,n=-\infty}^{\infty}q^{m^2+n^2},$$

which implies

$$\varphi^2(q)=\sum_{n=-\infty}^{\infty}\sum_{2r+3s=n}q^{r^2+s^2}.$$

We consider the thirteen cases for n, with $n=13m$, $13m\pm1$, $13m\pm2$, $13m\pm3$, $13m\pm4$, $13m\pm5$, and $13m\pm6$ by setting $(r,s)=(2m+3t,3m-2t)$, $(2m+3t-1,3m-2t+1)$, $(2m+3t+1,3m-2t-1)$, $(2m+3t+1,3m-2t)$, $(2m+3t-1,3m-2t)$, $(2m+3t,3m-2t+1)$, $(2m+3t,3m-2t-1)$, $(2m+3t+2,3m-2t)$, $(2m+3t-2,3m-2t)$, $(2m+3t+1,3m-2t+1)$, $(2m+3t-1,3m-2t-1)$, $(2m+3t,3m-2t+2)$, and $(2m+3t,3m-2t-2)$ respectively in the above, and find that

$$\begin{aligned}\varphi^2(q) = &\ 2\sum_{m,t=-\infty}^{\infty} q^{13m^2+13t^2+4m+6t+1} + 2\sum_{m,t=-\infty}^{\infty} q^{13m^2+13t^2+6m+4t+1}\\ &+ 2\sum_{m,t=-\infty}^{\infty} q^{13m^2+13t^2+2m+10t+2} + 2\sum_{m,t=-\infty}^{\infty} q^{13m^2+13t^2+10m+2t+2}\\ &+ 2\sum_{m,t=-\infty}^{\infty} q^{13m^2+13t^2+8m+12t+4} + 2\sum_{m,t=-\infty}^{\infty} q^{13m^2+13t^2+12m+8t+4}\\ &+ \sum_{m,t=-\infty}^{\infty} q^{13m^2+13t^2}.\end{aligned}$$

On transforming the q-series on the right side of the above in terms of Ramanujan's theta function $f(a,b)$, we find that

$$\begin{aligned}\varphi^2(q) = &\ \varphi^2(q^{13}) + 4qf(q^9,q^{17})f(q^7,q^{19})\\ &+ 4q^2f(q^{11},q^{15})f(q^3,q^{23}) + 4q^4f(q^5,q^{21})f(q,q^{25}).\end{aligned}$$

Changing q to $-q$ in the above identity and then dividing the resulting equation throughout by $\varphi^2(-q^{13})$, we arrive at (2.4). By the definitions of $E_i's$ and $\chi(q)$, we have

$$E_1E_2E_3E_4E_5E_6 = -q^7\frac{\chi(-q)}{\chi^{13}(-q^{13})}.$$

Squaring on both sides of the identity (2.4) and using the above in the resulting equation, we find that

$$\begin{aligned}&\frac{1}{E_1E_3} + \frac{1}{E_2E_6} + \frac{1}{E_4E_5}\\ &\quad = \frac{\chi^{13}(-q^{13})}{2q^7\chi(-q)}\left(E_1^2E_3^2 + E_2^2E_6^2 + E_4^2E_5^2 - \frac{1}{16}\left(\frac{\varphi^2(-q)}{\varphi^2(-q^{13})} - 1\right)^2\right).\end{aligned} \tag{2.6}$$

Changing q to q^{13}, then setting $(a,b,c,d) = (q^2,q^4,q^3,q^6)$ in Weierstrass' three-term identity (2.2), we have

$$\begin{aligned}&f(-q^2,-q^{11})f(-q^3,-q^{10})f(-q^4,-q^9)f(-q^6,-q^7)\\ &- f(-q,-q^{12})f(-q^4,-q^9)f(-q^5,-q^8)f(-q^6,-q^7)\\ &\quad = qf(-q,-q^{12})f(-q^2,-q^{11})f(-q^3,-q^{10})f(-q^5,-q^8).\end{aligned}$$

Now, dividing the above identity by

$$qf(-q,-q^{12})f(-q^2,-q^{11})f(-q^3,-q^{10})f(-q^4,-q^9)f(-q^5,-q^8)f(-q^6,-q^7),$$

we obtain

$$\frac{1}{\alpha} = \frac{1}{\beta} + \frac{1}{\gamma}, \tag{2.7}$$

where $\alpha = qf(-q, -q^{12})f(-q^5, -q^8)$, $\beta = qf(-q^2, -q^{11})f(-q^3, -q^{10})$, and $\gamma = f(-q^4, -q^9)f(-q^6, -q^7)$. From the above, it follows that

$$\frac{1}{\beta^3} + \frac{1}{\gamma^3} - \frac{1}{\alpha^3} = \frac{-3}{\alpha\beta\gamma}.$$

Multiplying (1.2) and (1.3), and then employing the above in the resulting identity, we find that

$$\frac{1}{\alpha\beta\gamma}(\gamma^3 + \beta^3 - \alpha^3) = 10 + 5\frac{f^2(-q)}{qf^2(-q^{13})} + \frac{f^4(-q)}{q^2 f^4(-q^{13})}. \tag{2.8}$$

Also, from the identity (2.7), we have

$$(\beta + \gamma - \alpha)^2 = \alpha^2 + \beta^2 + \gamma^2 \tag{2.9}$$

and

$$(\beta + \gamma - \alpha)^3 = \beta^3 + \gamma^3 - \alpha^3 + 3\alpha\beta\gamma. \tag{2.10}$$

Further, we observe that

$$\alpha\beta\gamma = q^2 f(-q)f^5(-q^{13}) \tag{2.11}$$

Efficiently employing the relations (2.9)–(2.11) on the left side of the identity (2.8), we obtain

$$\alpha^2 + \beta^2 + \gamma^2 = q^{\frac{2}{3}} f^2(-q)f^2(-q^{13}) \left(13q\frac{f^2(-q^{13})}{f^2(-q)} + 5 + \frac{f^2(-q)}{qf^2(-q^{13})}\right)^{\frac{2}{3}}. \tag{2.12}$$

Changing q to q^{26}, then setting $(a, b, c, d) = (q^{-6}, q^7, q^{-4}, q^9)$, $(q^{-5}, q^8, q^{-1}, q^{12})$, and $(q^{-3}, q^{10}, q^{-2}, q^{11})$ in (2.2) and on elementary simplification, respectively we find that

$$E_2^2 E_6^2 = E_1 E_3 + q\frac{\alpha^2(q^2)}{\varphi^4(-q^{13})},$$

$$E_4^2 E_5^2 = E_2 E_6 + q\frac{\beta^2(q^2)}{\varphi^4(-q^{13})},$$

and

$$E_1^2 E_3^2 = E_4 E_5 + q\frac{\gamma^2(q^2)}{\varphi^4(-q^{13})}.$$

Adding the above three identities and employing (2.4) and (2.12) in the resulting sum, we obtain

$$E_1^2E_3^2+E_2^2E_6^2+E_4^2E_5^2=$$
$$\frac{1}{4}\left(\frac{\varphi^2(-q)}{\varphi^2(-q^{13})}-1\right)+q^{\frac{7}{3}}\frac{f^2(-q^2)f^2(-q^{26})}{\varphi^4(-q^{13})}\left(13\frac{q^2f^2(-q^{26})}{f^2(-q^2)}+5+\frac{f^2(-q^2)}{q^2f^2(-q^{26})}\right)^{\frac{2}{3}}.$$

Employing the above in (2.6), we obtain the required result (2.5). This completes the proof of Theorem 2.3. □

Theorem 2.4. We have

$$1+H_1+H_2+H_3+H_4+H_5+H_6=\frac{\psi(q^{\frac{1}{13}})}{q^{\frac{21}{13}}\psi(q^{13})}, \tag{2.13}$$

$$H_1H_5+H_2H_3+H_4H_6=\frac{\psi^2(q)}{q^3\psi^2(q^{13})}-1, \tag{2.14}$$

and

$$\frac{1}{H_1H_5}+\frac{1}{H_2H_3}+\frac{1}{H_4H_6}=\frac{q^4}{2}\frac{f_2^3f_{26}^9}{f_1f_{13}^{11}}\left(\frac{\psi^2(q)}{q^3\psi^2(q^{13})}-6+5\frac{q^3\psi^2(q^{13})}{\psi^2(q)}\right)$$
$$-\frac{q^{\frac{5}{3}}}{2}\frac{f_1^3f_{26}^5}{f_2f_{13}^7}\left(13q\frac{f_{13}^2}{f_1^2}+5+\frac{f_1^2}{qf_{13}^2}\right)^{\frac{2}{3}}, \tag{2.15}$$

where

$$H_1=\frac{f(q,q^{12})}{q^{\frac{6}{13}}\psi(q^{13})},\ H_2=\frac{f(q^2,q^{11})}{q^{\frac{11}{13}}\psi(q^{13})},\ H_3=\frac{f(q^3,q^{10})}{q^{\frac{15}{13}}\psi(q^{13})},$$
$$H_4=\frac{f(q^4,q^9)}{q^{\frac{18}{13}}\psi(q^{13})},\ H_5=\frac{f(q^5,q^8)}{q^{\frac{20}{13}}\psi(q^{13})},\ \textit{and}\ H_6=\frac{f(q^6,q^7)}{q^{\frac{21}{13}}\psi(q^{13})}.$$

Proof. We have [2, p. 49]

$$\psi(q)=\frac{1}{2}\sum_{r=0}^{n-1}q^{r(r-1)/2}f(q^{n(n-2r+1)/2},q^{n(n+2r-1)/2}). \tag{2.16}$$

Setting $n=13$ in the above, then changing q to $q^{\frac{1}{13}}$, and dividing the resulting identity by $\psi(q^{13})$, we arrive at (2.13). By the definition of $\psi(q)$, we have

$$\psi^2(q)=\sum_{m,n=-\infty}^{\infty}q^{2m^2+2n^2-m-n},$$

which implies

$$\psi^2(q)=\sum_{n=-\infty}^{\infty}\sum_{2r+3s=n}q^{2r^2+2s^2-r-s}.$$

We consider the thirteen cases for n, with $n=13m$, $13m\pm1$, $13m\pm2$, $13m\pm3$, $13m\pm4$, $13m\pm5$, and $13m\pm6$ by setting the values for (r,s) as in the proof (2.4) on the right hand side series of the above, we find that

$$\begin{aligned}\psi^2(q)=&\sum_{m,t=-\infty}^{\infty}q^{26m^2+26t^2-5m-t}+\sum_{m,t=-\infty}^{\infty}q^{26m^2+26t^2-m-21t+4}\\&+\sum_{m,t=-\infty}^{\infty}q^{26m^2+26t^2-9m+19t+4}+\sum_{m,t=-\infty}^{\infty}q^{26m^2+26t^2+3m+11t+1}\\&+\sum_{m,t=-\infty}^{\infty}q^{26m^2+26t^2-13m-13t+3}+\sum_{m,t=-\infty}^{\infty}q^{26m^2+26t^2+7m-9t+1}\\&+\sum_{m,t=-\infty}^{\infty}q^{26m^2+26t^2-17m+7t+3}+\sum_{m,t=-\infty}^{\infty}q^{26m^2+26t^2+11m+23t+6}\\&+\sum_{m,t=-\infty}^{\infty}q^{26m^2+26t^2-21m-25t+10}+\sum_{m,t=-\infty}^{\infty}q^{26m^2+26t^2+15m+3t+2}\\&+\sum_{m,t=-\infty}^{\infty}q^{26m^2+26t^2-25m-5t+6}+\sum_{m,t=-\infty}^{\infty}q^{26m^2+26t^2+19m-17t+6}\\&+\sum_{m,t=-\infty}^{\infty}q^{26m^2+26t^2-29m+15t+10}.\end{aligned}$$

By combining the Entries 30(ii) and 30(iii) of Chap. 16 of Ramanujan's second notebook [7], we obtain

$$f(a,b)=f(a^3b,ab^3)+af\left(\frac{b}{a},\frac{a}{b}a^4b^4\right). \tag{2.17}$$

On transforming the q-series on the right side of the above in terms of Ramanujan's theta function $f(a,b)$, and then efficiently employing (2.17) in the resulting identity, we find that

$$\begin{aligned}\psi^2(q)=q^3\psi^2(q^{13})&+qf(q,q^{12})f(q^5,q^8)\\&+qf(q^2,q^{11})f(q^3,q^{10})+f(q^4,q^9)f(q^6,q^7).\end{aligned}$$

Dividing the above equation throughout by $q^3\psi^2(q^{13})$, we obtain (2.14). By the definitions of $H_i's$ and $\chi(q)$, we observe that

$$H_1H_2H_3H_4H_5H_6 = \frac{\chi^{13}(-q^{13})}{q^7\chi(-q)}.$$

Squaring on both sides of the identity (2.14) and using the above relation, we find that

$$\begin{aligned}&\frac{1}{H_1H_5}+\frac{1}{H_2H_3}+\frac{1}{H_4H_6}\\&=\frac{q^7\chi(-q)}{2\chi^{13}(-q^{13})}\left(\left(\frac{\psi^2(-q)}{\psi^2(-q^{13})}-1\right)^2-(H_1^2H_5^2+H_2^2H_3^2+H_4^2H_6^2)\right).\end{aligned}\tag{2.18}$$

Changing a to $-a$, c to $-c$, and q to q^{13} in (2.2), then setting $(a,b,c,d) = (q^{-6},q^7,q^{-4},q^9)$, $(q^{-5},q^8,q^{-1},q^{12})$ and $(q^{-3},q^{10},q^{-2},q^{11})$ in the resulting identity respectively, we find that

$$H_1^2H_5^2 = H_4H_6+\frac{\alpha^2(q)}{q^6\psi^4(q^{13})},$$

$$H_2^2H_3^2 = H_1H_5+\frac{\beta^2(q)}{q^6\psi^4(q^{13})},$$

and

$$H_4^2H_6^2 = H_2H_3+\frac{\gamma^2(q)}{q^6\psi^4(q^{13})}.$$

Adding the above three identities and employing (2.14) and (2.12) in the resulting sum, we obtain that

$$\begin{aligned}&H_1^2H_5^2+H_2^2H_3^2+H_4^2H_6^2 =\\&\quad 4\left(\frac{\psi^2(q)}{q^3\psi^2(q^{13})}-1\right)+\frac{f^2(-q)f^2(-q^{13})}{q^{\frac{16}{3}}\psi^4(q^{13})}\left(13\frac{f^2(-q^{13})}{qf^2(-q)}+5+\frac{f^2(-q)}{q^2f^2(-q^{13})}\right)^{\frac{2}{3}}.\end{aligned}$$

Employing the above in (2.18), we obtain our required result (2.15). This completes the proof of Theorem 2.4. □

Theorem 2.5. We have

$$\begin{aligned}\frac{\varphi^4(-q)}{\varphi^4(-q^{13})} = 1+8\Bigg(&\frac{E_1}{H_3^2(q^2)E_4^2(-q)}+\frac{E_2}{H_4^2(q^2)E_3^2(-q)}+\frac{E_3}{H_2^2(q^2)E_5^2(-q)}\\&+\frac{E_4}{H_5^2(q^2)E_2^2(-q)}+\frac{E_5}{H_1^2(q^2)E_6^2(-q)}+\frac{E_6}{H_6^2(q^2)E_1^2(-q)}\Bigg),\end{aligned}\tag{2.19}$$

where $E_i = E_i(q)$'s and $H_i = H_i(q)$'s are as defined in Theorem 2.3 and 2.4.

Proof. We have [7, Chap. 17, Entry 8(ii), p. 209]

$$\varphi^4(q) = 1 + 8\sum_{n=1}^{\infty} \frac{nq^n}{1+(-q)^n}. \tag{2.20}$$

From the above, we have

$$\varphi^4(-q) = 1 + 8\sum_{n=1}^{\infty} \frac{q^{2n}}{(1+q^{2n})^2} - 8\sum_{n=0}^{\infty} \frac{q^{2n+1}}{(1+q^{2n+1})^2}. \tag{2.21}$$

We observe that

$$\sum_{n=0}^{\infty} \frac{q^{2n+1}}{(1+q^{2n+1})^2} = \sum_{n=0}^{\infty}\sum_{r=0}^{12} \frac{q^{26n+2r+1}}{(1+q^{26n+2r+1})^2},$$

which implies that

$$\sum_{n=0}^{\infty} \frac{q^{2n+1}}{(1+q^{2n+1})^2} - \sum_{n=0}^{\infty} \frac{q^{26n+13}}{(1+q^{26n+13})^2} = \sum_{n=-\infty}^{\infty}\sum_{r=1}^{6} \frac{q^{26n+2r-1}}{(1+q^{26n+2r-1})^2}. \tag{2.22}$$

Similarly, we find that

$$\sum_{n=0}^{\infty} \frac{q^{2n}}{(1+q^{2n})^2} - \sum_{n=0}^{\infty} \frac{q^{26n}}{(1+q^{26n})^2} = \sum_{n=-\infty}^{\infty}\sum_{r=1}^{6} \frac{q^{26n+2r}}{(1+q^{26n+2r})^2}. \tag{2.23}$$

Using (2.21)–(2.23), we have

$$\varphi^4(-q) - \varphi^4(-q^{13}) = 8\sum_{n=-\infty}^{\infty}\sum_{r=1}^{6} \left(\frac{q^{26n+2r}}{(1+q^{26n+2r})^2} - \frac{q^{26n+2r-1}}{(1+q^{26n+2r-1})^2} \right). \tag{2.24}$$

Changing q to q^{26} and then setting $a = -q^{2r}$ and $b = -q^{13-2r}$ in the Bailey's summation formula (2.1), and using the resulting identity on the right side of (2.24), we obtain

$$\varphi^4(-q) - \varphi^4(-q^{13}) = 8f^6(-q^{26}) \sum_{r=1}^{6} \frac{q^{2r} f(-q^{13}, -q^{13}) f(-q^{13-4r}, -q^{13+4r})}{f^2(q^{2r}, q^{26-2r}) f^2(q^{13-2r}, q^{13+2r})}.$$

Finally, dividing the above equation throughout by $\varphi^4(-q^{13})$, we obtain the required result (2.19). This completes the proof of Theorem 2.5. $\square$

Theorem 2.6. We have

$$\frac{\psi^4(q)}{q^6\psi^4(q^{13})} = 1 + \frac{H_1}{E_1^2} + \frac{H_3}{E_2^2} + \frac{H_5}{E_3^2} + \frac{H_6}{E_4^2} + \frac{H_4}{E_5^2} + \frac{H_2}{E_6^2}, \tag{2.25}$$

where $E_i = E_i(q)$'s and $H_i = H_i(q)$'s are as defined in Theorem 2.3 and 2.4.

Proof. We have [7, Chap. 16, Entry 25(vii), p. 198]

$$16q\psi^4(q^2) = \varphi^4(q) - \varphi^4(-q), \tag{2.26}$$

from which it follows that

$$\begin{aligned}16\big(q\psi^4(q^2) - q^{13}\psi^4(q^{26})\big) &= \varphi^4(q) - \varphi^4(q^{13}) - \big(\varphi^4(-q) - \varphi^4(-q^{13})\big)\\ &= 8f^6(-q^{26})\sum_{r=1}^{6}\frac{q^{2r}f(q^{13},q^{13})f(q^{13-4r},q^{13+4r})}{f^2(q^{2r},q^{26-2r})f^2(-q^{13-2r},-q^{13+2r})}\\ &\quad - 8f^6(-q^{26})\sum_{r=1}^{6}\frac{q^{2r}f(-q^{13},-q^{13})f(-q^{13-4r},-q^{13+4r})}{f^2(q^{2r},q^{26-2r})f^2(q^{13-2r},q^{13+2r})},\end{aligned}$$

which implies that

$$\begin{aligned}&2\big(q\psi^4(q^2) - q^{13}\psi^4(q^{26})\big)\\ &= f^6(-q^{26})\sum_{r=1}^{6}\frac{q^{2r}A(q,r)}{f^2(q^{2r},q^{26-2r})f^2(-q^{13-2r},-q^{13+2r})f^2(q^{13-2r},q^{13+2r})},\end{aligned}$$

where

$$\begin{aligned}A(q,r) &= f(q^{13},q^{13})f(q^{13-4r},q^{13+4r})f^2(q^{13-2r},q^{13+2r})\\ &\quad - f(-q^{13},-q^{13})f(-q^{13-4r},-q^{13+4r})f^2(-q^{13-2r},-q^{13+2r}).\end{aligned}$$

Changing a to $-a$, c to $-c$, and q to q^{26}, then setting $a = q^{2r}$, $b = 1$, $c = q^{13-2r}$, and $d = q^{13-2r}$ in (2.2), and using the resulting equation on the right hand side of the above equation, we obtain

$$\begin{aligned}&q\psi^4(q^2) - q^{13}\psi^4(q^{26})\\ &= f^6(-q^{26})\psi(q^{26})\sum_{r=1}^{6}\frac{q^{13-2r}f(q^{4r},q^{26-4r})}{f^2(q^{13-2r},q^{13+2r})f^2(-q^{13-2r},-q^{13+2r})},\end{aligned} \tag{2.27}$$

From [7, Chap. 16, Entry 30(iv), p. 199], we have

$$f(a,b)f(-a,-b) = 2af(-a^2,-b^2)\varphi(-ab). \tag{2.28}$$

Employing the identity on the right hand side of (2.27), then dividing throughtout by $q^{13}\psi^4(q^{26})$ and finally replacing q to $q^{1/2}$ in the resulting equation, we obtain (2.25). This completes the proof of Theorem 2.6. $\square$

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