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Monodromy of local systems on the complement of line arrangements

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Abstract. We show that a previous construction for fibered arrangements can be generalized to obtain new algebraic complexes computing the local system (co)homology of any (affine) line arrangement. We obtain a description of the characteristic variety as the locus where the monodromies around the singular points of the arrangement share a common fixed subspace. We apply this construction to the problem of deciding whether the monodromy action on the homology of the Milnor fiber is trivial. In particular, we improve a previous result motivated by the a-monodromicity conjecture stated by the second author.

Keywords: Hyperplane Arrangements, Characteristic varieties, Milnor fiber

MSC 2020 classification: 32S22 55N25 32S50

1 Introduction

A standard result in the theory of hyperplane arrangements is that the cohomology of the complement with trivial coefficients depends only on the combinatorics of the arrangement (see [20]). Similar questions can be posed for cohomology with local coefficients, defined by a representation of the fundamental group of the complement over a given ring. In particular, one is interested in abelian rank-one local systems, defined by representations of the fundamental group over $\mathbb{C}$; such representations automatically factor through the first homology group of the complement.

Much recent work has been devoted to the study of the jump loci of cohomology with local coefficients, which give rise to the so-called *characteristic* and *resonance varieties* (see for example [2], [4], [6], [7], [8], [9], [10], [11], [12], [13], [14], [16], [17], [19], [25], [27], [30], [32]). By general results (see [1]), the characteristic variety is a union of subtori in the parameter space; however, it remains an open question whether it depends only on the combinatorics.

In this paper we generalize some results obtained in [23], where fibered arrangements were considered. In that work, explicit formulas were derived for the monodromies acting on the homology of the fibers around the vertical lines containing the singular points. The characteristic variety was described as the set of parameters for which the monodromies share a common left eigenvector. As an application, a sequence of arrangements whose characteristic varieties possess a translated one-dimensional component was constructed. The first arrangement in this sequence is the well-known example originally discovered in [27]. It is known that the "mysterious" part of the characteristic variety consists of translated tori, which are not known to depend uniquely on the combinatorics (see, for example, [11], [8]).

Paper [23] also contains the construction of an algebraic complex computing local homology for fibered arrangements, which differs from other known complexes.

Here we generalize this algebraic complex to one computing local homology in general, that is, for arrangements that are not necessarily fibered. We make use of an exact sequence arising from a kind of "deletion–contraction" lemma for local systems (see [5]). Our main result identifies the characteristic variety of the arrangement $\mathcal{A}$:

$$V_1(\mathcal{A}) = \{(\rho_\ell)_{\ell\in\mathcal{A}} \in (\mathbb{C}^*)^r : \ \dim(H_1(\mathcal{M}(\mathcal{A}); \mathcal{L}_\rho)) > 0\}$$

(where $r = \#\mathcal{A}$) with the locus of parameters for which the monodromy operators around the vertical lines containing singular points have a common (left) fixed subspace.

Beyond the intrinsic interest of the previous result, there is a strong relationship with the (co-)homology of the associated Milnor fiber F. Indeed, it is a standard result that the homology of F with coefficients in a ring A, viewed as a module under the monodromy action, coincides with the local homology of the complement $\mathcal{M}(\mathcal{A})$ with coefficients in the Laurent polynomial ring $R = A[s^{\pm 1}]$, where the monodromy corresponds to multiplication by s.

The homology of the Milnor fiber remains mysterious in general; even a formula for the first Betti number is not known.

In [24], a line arrangement $\mathcal{A}$ is called a-monodromic if the monodromy of its Milnor fiber is trivial (here we take $A = \mathbb{Q}$). This is equivalent to the condition that the $\mathbb{Q}[s, s^{-1}]$-module $H_*(\mathcal{M}(\mathcal{A}), \mathbb{Q}[s, s^{-1}])$ has only $(s-1)$ as elementary divisors (see part 4 below). Moreover, $\mathcal{A}$ is a-monodromic if and only if the first Betti number of the Milnor fiber F is minimal.

It is not clear whether this property is combinatorial. In this direction, in [24] it was conjectured that if the graph of double points of $\mathcal{A}$ is connected, then $\mathcal{A}$ is a-monodromic, and this conjecture was proved under certain additional assumptions (see also [29], where many related results and open problems are listed).

Here we use the complex constructed below to prove that the conjecture holds under further conditions on the graph of double points, which are weaker than those imposed in [24].

In Part 2 we recall the constructions from [23], adding some remarks and proofs.

In Part 3 we generalize these constructions to arbitrary line arrangements to obtain the above-mentioned description of the characteristic variety.

In Part 4 we review some properties of the Milnor fiber (following [24]), restating the conjectures concerning a-monodromicity.

In Part 5 we establish general properties of our complex and extend the a-monodromicity result to the case of an *inductive* graph of double points.

Our construction appears particularly promising for the general study of the homology of the Milnor fiber of an arrangement.

2 The case of fibered arrangements

We start from some basic notation and constructions.

We consider an affine line arrangement $\mathcal{A}$ in $\mathbb{C}^2$. We assume $\mathcal{A}$ is defined over $\mathbb{R}$. Denote by $\mathcal{M}(\mathcal{A}) = \mathbb{C}^2 \setminus \bigcup_{H\in\mathcal{A}} H$ the complement of the arrangement in $\mathbb{C}^2$. We will use affine coordinates (x, y), $x, y \in \mathbb{C}$. Let $\pi\colon \mathbb{C}^2 \to \mathbb{C}$ be the projection $\pi(x, y) = x$ onto the x-axis and $\pi' = \pi_{|\mathcal{M}(\mathcal{A})}\colon \mathcal{M}(\mathcal{A}) \to \mathbb{C}$ its restriction. Let $\mathcal{S} \subset \mathbb{C}^2$ be the set of singular points of the arrangement, $\pi(\mathcal{S}) \subset \mathbb{C}$ its projection and $B = \mathbb{C} \setminus \pi(\mathcal{S})$. With our assumptions $\mathcal{S} \subset \mathbb{R}^2 \subset \mathbb{C}^2$ and $\pi(\mathcal{S})$ is contained in the real axis $\mathbb{R}$ of $\mathbb{C}$.

A line $\ell \in \mathcal{A}$ will be called *horizontal* if $\pi_{|\ell}$ is surjective, otherwise it will be called *vertical*. We order the arrangement

$$\mathcal{A} = \{\ell_1, \dots \ell_n, \ell'_1, \dots, \ell'_m\}$$

so that the first n lines are horizontal, and the remaining m lines are vertical. Notice that $\ell'_j = \pi^{-1}(\xi)$ for some $\xi \in \pi(\mathcal{S})$, while $\pi(\ell_i) = \mathbb{C}$.

One has a fiber bundle structure

$$\pi'' = \pi'_{|(\pi')^{-1}(B)}\colon (\pi')^{-1}(B) \to B \tag{2.1}$$

where the fibers $F_x := (\pi')^{-1}(x)$, $x \in B$, are given by a copy of $\mathbb{C}$ with n points removed. Notice also that if $\forall \xi \in \pi(\mathcal{S})$ the vertical line $\pi^{-1}(\xi)$ belongs to $\mathcal{A}$, then all the complement $\mathcal{M}(\mathcal{A})$ is a fiber bundle over B.

Let $\mathbb{C}_x = \pi^{-1}(x)$, $x \in B$; then $F_x = \mathbb{C}_x \setminus \{p_1^{(x)}, \dots, p_n^{(x)}\}$ where the points $p_i^{(x)} = (x, y_i^{(x)})$ are the intersections of the horizontal lines ℓ_j with $\mathbb{C}_x$. Notice that $y_i^{(x)} \in \mathbb{R}$; we assume that $y_1^{(x)} < \dots < y_n^{(x)}$.

In order to make cohomological computations, we fix a standard fiber $F_0 = (\pi'')^{-1}(x_0)$, where the point x_0 lies on the real axis of $\mathbb{C}$ and inside B. Then $F_0 = \mathbb{C}_{x_0} \setminus \{p_1^0, \dots, p_n^0\}$, where we set $p_i^0 = p_i^{(x_0)} = (x_0, y_i^{(x_0)}),\ i = 1, \dots, n$. Notice that all points $y_i^{(x_0)}$ lie on the real axis $\mathbb{R}_{x_0} := \Re(\mathbb{C}_{x_0})$ of $\mathbb{C}_{x_0}$. We assume that the indices of the lines are taken in increasing order of their intersections with the real axis $\mathbb{R}_{x_0}$ (canonically oriented).

We also take a basepoint $P_0 \equiv (x_0, y_0) \in \mathbb{C}_{x_0}$, $P_0 \in F_0$. We choose $\Im(y_0) \gg 0$, and elementary well-ordered generators $\alpha_1, \dots, \alpha_n$ of $\pi_1(F_0)$. In a standard way, α_i is constructed by a segment connecting P_0 to a small circle centered in p_i^0, turning around p_i^0 counterclockwise.

The same construction is performed on any fiber F_x, $x \in B \cap \mathbb{R}$, by using the basepoint $P_0^{(x)} \equiv (x, y_0)$ and generators $\alpha_i^{(x)}$ analogous to the α_i; the indices here are locally computed according to the growing intersections of the lines with $\Re(\mathbb{C}_x)$.

Recall that an abelian rank-1 $\mathbb{C}$-local system on $\mathcal{M}(\mathcal{A})$ is determined by a one dimensional representation of $\pi_1(\mathcal{M}(\mathcal{A}))$. It consists of taking $\mathbb{C}$ as an $H_1(\mathcal{M}(\mathcal{A}); \mathbb{Z})$-module, where the action is given by mapping standard generators, (small canonically oriented circles $\beta_i,\ i = 1, \dots, n+m$, around the lines in $\mathcal{A}$,) to multiplication by non-zero numbers. Therefore any such local system corresponds to a map $\mathcal{A} \to \mathbb{C}^*$. After choosing an ordering of $\mathcal{A}$, we can see it as an $(n+m)$-tuple of parameters in $(\mathbb{C}^*)^{n+m}$.

We call $s_1, \dots, s_n$ the parameters (in $\mathbb{C}^*$) which correspond to the lines $\ell_1, \dots, \ell_n$ and $t_1, \dots, t_m$ those corresponding to the lines $\ell'_1, \dots, \ell'_m$ respectively. Denote by $\mathcal{L} = \mathcal{L}_{s,t}$ the corresponding local system. The restriction $\mathcal{L}_{|_{F_0}}$ is a local system which is clearly determined solely by the only parameters s_i. We write $\mathcal{L}_s := \mathcal{L}_{|_{F_0}}$.

In [23] $\mathcal{M}(\mathcal{A})$ was assumed to fiber over B. This amounts to the singular set $\mathcal{S}$ being contained into the union of the vertical lines.

For the rest of this section we assume that $\mathcal{M}(\mathcal{A})$ fibers over B. We review and clarify some computations found in [23].

Lemma 1. *We have*

$$H_1(\mathcal{M}(\mathcal{A}); \mathcal{L}_{s,t}) = H_0(B; H_1(F_0; \mathcal{L}_s)) \oplus H_1(B; H_0(F_0; \mathcal{L}_s)) \tag{2.2}$$

where the second factor on the right-hand side vanishes if some of the s_i is different from 1. $\boxed{QED}$

Proof. By the spectral sequence for fibered spaces (see [15, Section 4.17]) one has

$$E^2_{p,q} \;=\; H_p(B; H_q(\{F_x\}; \mathcal{L}_{s,t})).$$

Since both B and F_0 are homotopy equivalent to $1-$ complexes, $E^2_{2,0} = E^2_{0,2} = 0$, so

$$H_1(\mathcal{M}(\mathcal{A}); \mathcal{L}_{s,t}) = H_0(B; H_1(\{F_x\}; \mathcal{L}_{s,t})) \oplus H_1(B; H_0(\{F_x\}; \mathcal{L}_{s,t})).$$

This reduces to (2.2) because

$$H_*(\{F_x\}; \mathcal{L}_{s,t}) \cong H_*(F_0; (\mathcal{L}_{s,t})_{|F_0}) \cong H_*(F_0; \mathcal{L}_s).$$

For non-trivial coefficients the 0-th local homology group $H_0(F_0; \mathcal{L}_s)$ vanishes (see also [26]). $\boxed{QED}$

In this paper we consider *global components* of the characteristic variety, meaning components not contained in any coordinate hypertorus $s_i = 1$. Therefore, $s_i \neq 1, \forall i$, and in particular $\mathcal{L}_s$ is non trivial. Therefore, it remains to compute $H_0(B; H_1(F_0; \mathcal{L}_s))$, where $H_1(F_0; \mathcal{L}_s)$ is a $\pi_1(B)$-module.

We will use several times the following remarks.

Remark 1. Let $S = \mathbb{C} \setminus \{p_1, \ldots, p_n\}$, $P \in S$ a basepoint, and $\lambda_1, \ldots, \lambda_n$ a well ordered set of elementary generators of $\pi_1(S, P)$, where λ_i encircles the point p_i, $i = 1, \ldots, n$. Let $\mathcal{L}$ be a rank-1 local system on S, given by a map $\pi_1(S, P) \to \mathbb{C}^*$, taking the class of the oriented generator λ_i to $u_i \in \mathbb{C}^*$ ($i = 1, \ldots, n$). Then the algebraic complex

$$0 \longrightarrow \oplus_{i=1}^n \mathbb{C}[\lambda_i] \xrightarrow{\partial} \mathbb{C}[P] \longrightarrow 0 \qquad (2.3)$$

where $\partial([\lambda_i]) = (u_i - 1).[P]$, $i = 1, \ldots, n$, calculates the local system homology, i.e.

$$H_1(S; \mathcal{L}) = \ker(\partial);\ H_0(S; \mathcal{L}) = \mathrm{coker}(\partial).$$

The remark follows directly from the fact that S deformation retracts over the wedge of 1-spheres $\bigcup_{i=1}^n \lambda_i$ and by the combinatorial computation of the local homology (see [26]).

From remark 1 applied when $S = F_0$, $p_i = \ell_i \cap F_0$, $u_i = s_i$, $i = 1, \ldots, n$, one has directly

Lemma 2. *Assuming $\mathcal{L}_s$ non-trivial one has* $\dim(H_1(F_0; \mathcal{L}_s)) = n - 1$. *A basis when all s_i are different from 1 is given by the classes of the 1-cycles*

$$\tilde{\alpha}_{i,i+1} = (1 - s_{i+1})[\alpha_i] - (1 - s_i)[\alpha_{i+1}]. \qquad (2.4)$$

$\boxed{QED}$

Remark 2. In the hypotheses of remark 1, let $\varphi : \mathbb{Z}[F_n] \to A$, where we set $A := Im(\varphi) = \mathbb{Z}[u_1^{\pm 1}, \dots, u_n^{\pm 1}] \subset \mathbb{C}$, be the homomorphism defined on the group algebra of the free group F_n generated by $\lambda_1, \dots, \lambda_n$ by sending λ_i to u_i. It follows from free calculus (thm. 3.6 in [3]) that $\sum_{i=1}^n v_i(u_i - 1) = 0$, $v_i \in A$, iff there exists $\lambda \in F_n$, $\varphi(\lambda) = 0$ such that $\varphi(\frac{\partial}{\partial \lambda_i}(\lambda)) = v_i$.
By applying this to the local system $\mathcal{L}_s$ on F_0, one has that

$$\sum_{i=1}^n v_i(s_i - 1) = \sum_{i=1}^n v_i \partial([\alpha_i]) \ = \ 0 \tag{2.5}$$

iff there exists

$$\alpha \in Ker(\varphi : F_n[\alpha_1, \dots, \alpha_n] \to \mathbb{Z}[s_1^{\pm 1}, \dots, s_n^{\pm 1}])$$

such that $\varphi\left(\frac{\partial}{\partial \alpha_i}(\alpha)\right) = v_i$. One has

$$\alpha - 1 = \sum_{i=1}^n \left(\frac{\partial}{\partial \alpha_i}(\alpha)\right)(\alpha_i - 1)$$

in the group algebra of $F_n[\alpha_1, \dots, \alpha_n]$. We write

$$[\alpha] := \sum_{i=1}^n \varphi\left(\frac{\partial}{\partial \alpha_i}(\alpha)\right)[\alpha_i] \tag{2.6}$$

as the 1-cycle determined by α.

Notice that the element $\tilde{\alpha}_{i,i+1}$ in (2.4) is given by the commutator $\tilde{\alpha}_{i,i+1} = [\alpha_i \alpha_{i+1} \alpha_i^{-1} \alpha_{i+1}^{-1}]$.

The braid group Br_n acts on $H_1(S; \mathcal{L})$. We identify here Br_n with the group of homotopy classes of compactly supported homeomorphism of $\mathbb{C}$ which preserve the set $\{p_1, \dots, p_n\}$. We will use that the action of $\beta \in Br_n$ on a 1-cycle of the form $[\alpha]$, $\alpha \in F_n \cap Ker(\varphi)$, is just

$$\beta.[\alpha] \ = \ [\beta.\alpha]$$

where on the right one has the induced action on the free group.

Since we suppose that all $s_i \neq 1$, it is convenient to introduce the generators

$$\alpha_{i,i+1} = \frac{\tilde{\alpha}_{i,i+1}}{(1 - s_i)(1 - s_{i+1})} = \frac{[\alpha_i]}{1 - s_i} - \frac{[\alpha_{i+1}]}{1 - s_{i+1}}. \tag{2.7}$$

We now use again remark 1 to compute $H_0(B; H_1(F_0; \mathcal{L}_s))$. We need a well-ordered set of generators of $\pi_1(B, x_0)$. We assume here $x_0 \in \mathbb{R}$, $x_0 \gg 0$, where

here $\mathbb{R}$ is the real axis of the first coordinate. Let $\pi(\mathcal{S}) = \{x_1, \ldots, x_m\} \subset \mathbb{R}$, where $x_1 > \cdots > x_m$, be the projection of the singular points of $\mathcal{A}$.

For any two points $x, x' \in \mathbb{R} \setminus \pi(\mathcal{S})$, we denote by $\gamma(x, x')$ a path contained in B, which connects x to x'. We require $\gamma(x, x')$ to be contained in the real axis, except that it avoids the projections of singular points by turning around them counterclockwise. More precisely, we fix a small ϵ, with $\epsilon < dist(\{x, x'\}, \pi(\mathcal{S}))$, where $dist$ is the Euclidean distance in $\mathbb{C}$. We require:

- $\Re(\gamma(x, x')(t)) = (1-t)x + tx',\ t \in [0,1]$
- if $|\Re(\gamma(x, x')(t)) - x_i| \geq \epsilon,\ i = 1, \ldots, m$, then $\Im(\gamma(x, x')(t)) = 0$
- if $\Re(\gamma(x, x')(\tau)) = x_i,\ \tau \in (0,1)$, then $\Im(\gamma(x, x')(t))(x - x_i) > 0$ for $t \in (\tau-\epsilon, \tau+\epsilon)$ (i.e., $\gamma(x, x')$ turns around the point x_i counterclockwise).

We have an induced transport isomorphism

$$\gamma(x, x')_* \colon H_1(F_x; \mathcal{L}_x) \to H_1(F_{x'}; \mathcal{L}_{x'}) \tag{2.8}$$

Precisely, the pull-back $\gamma(x, x')^*$ of the bundle (2.1) is trivial over the interval $[0,1]$. We have a homeomorphism $F : [0,1] \times F_x \to \gamma(x, x')^*$: we choose a compactly supported trivialization, meaning that $F(t, y) = (\gamma(x, x')(t), y)$ when $|y| \gg 0$. In particular, the restriction of F to the fibers takes the basepoint $P_0^{(x)}$ to the basepoint $P_0^{(x')}$.

For each $x_k \in \pi(\mathcal{S})$, we denote by $x'_k = x_k + \epsilon,\ x''_k = x_k - \epsilon \in \mathbb{R}$ two points very close to x_k and lying on opposite sides of x_k on the real axis of $\mathbb{C}$. We will use a well ordered set of generators of $\pi_1(B, x_0)$ given by the elementary paths

$$\gamma_k \;=\; \gamma(x_0, x'_k)\gamma(x'_k, x''_k)\gamma(x''_k, x'_k)(\gamma(x_0, x'_k))^{-1} \tag{2.9}$$

which encircle the points $x_k \in \pi(\mathcal{S})$, $k = 1, \ldots, m$.

Set $\alpha_{i,i+1}^{(x)} \in H_1(F_x; \mathcal{L}_x)$, $i = 1, \ldots, n-1$, the generators constructed as those in (2.7), by using the $\alpha_i^{(x)}$. Let also, for $x \neq x'$, $\sigma = \sigma(x, x')$ be the permutation of the indices $1, \ldots, n$ which is obtained as follows: the i-th line in the x-ordering (considering the growing intersections with $\Re(\mathbb{C}_x)$) is the $\sigma(i)$-th line in the x'-ordering. For each $i = 1, \ldots, n-1$ we set $\sigma(i, i+1) = +1$ or -1 depending on whether $\sigma(i+1) > \sigma(i)$ or $\sigma(i+1) < \sigma(i)$. In other words, $\sigma(i, i+1) = 1$ $(= -1)$ according whether the two lines of indices $i,\ i+1$ (in the x-ordering) do not intersect (intersect) in $\pi^{-1}([x, x'])$.

In order to clarify the indices of the parameters, we also write $s(\ell)$ as the parameter associated to $\ell \in \mathcal{A}$. So $s(\ell) = s_i$ when $\ell = \ell_i$ in the ordering of the intersections with F_0.

We also use $s_i^{(x)}$ to denote $s(\ell)$, for the line ℓ which intersects F_x in the i-th position in the x- ordering.

Let also $\mathcal{A}^{(x)}$ be the arrangement in $\Re(\mathbb{C}_x)$ induced by the intersections $p_1^{(x)}, \ldots, p_n^{(x)} \in \Re(\mathbb{C}_x)$ of $\mathbb{C}_x$ with $\mathcal{A}$. We write $C_i^{(x)} = (p_i^{(x)}, p_{i+1}^{(x)})$ for the chamber in $\mathcal{A}^{(x)}$ determined by two consecutive points ($i = 1, \ldots, n-1$). Let $C_0^{(x)} = (-\infty, p_1^{(x)})$ be the "lowest" unbounded chamber in $\mathcal{A}^{(x)}$. We write also $\mathcal{H}(C, C')$ as the set of horizontal lines separating (in $\mathbb{R}^2$) the two chambers C, C' (which can lie in different fibers).

The basic computation is given by the following formula.

Theorem 1 (Parallel transport)**.** *We have*

$$\begin{aligned}\gamma(x,x')_*(\alpha_{i,i+1}^{(x)}) &= \sigma(i,i+1) \sum_{j=m(i,i+1)}^{M(i,i+1)-1} \Big(\prod_{\substack{\ell \in \mathcal{H}(C_i^{(x)}, C_j^{(x')}) \\ \ell \notin \mathcal{H}(C_i^{(x)}, C_0^{(x)})}} s(\ell) \Big) \alpha_{j,j+1}^{(x')} = \\ &= \sigma(i,i+1) \sum_{j=m(i,i+1)}^{M(i,i+1)-1} \Big(\prod_{\substack{k>i \\ \sigma(k)\le j}} s_k^{(x)} \Big) \alpha_{j,j+1}^{(x')}\end{aligned} \tag{2.10}$$

where $m(i,i+1) = \min(\sigma(i), \sigma(i+1))$ and $M(i,i+1) = \max(\sigma(i), \sigma(i+1))$.

Proof. We use the above remark 2. First, assume the path $\gamma = \gamma(x_i', x_i'')$ runs over a semi-circle around one $x_i \in \pi(\mathcal{S})$, starting in x_i' and ending in x_i'', in counterclockwise direction. Let $i_1 < \cdots < i_n$ be the ordering of the horizontal lines induced by their intersection with the oriented line $\mathbb{R}_{x_i'}$ (therefore $k = \sigma(x_0, x_i')(i_k)$ in the above notations). Let $P_1, \ldots, P_h \in \mathbb{C}_{x_i}$ be the singular points of the arrangement which lie above x_i, of multiplicity respectively $\mu_1, \ldots, \mu_h (\geq 2)$. It is convenient to consider only the horizontal lines through P_j, so we set $m_j = \mu_j - 1$, $j = 1, \ldots, h$. Each P_j is the intersection of the vertical line ℓ_i' with consecutive horizontal lines $\ell_{i_{a_j}} \cap \cdots \cap \ell_{i_{a_j + m_j - 1}}$, where $a_j \in \{1, \ldots, n\}$. In other terms, the set $\{1, \ldots, n\}$ is partitioned into blocks $\Delta_1 \sqcup \cdots \sqcup \Delta_h$, where $\Delta_j = \{a_j, \ldots, a_j + m_j - 1\}$ (see fig. 1). Notice that $|\Delta_j| = 1$ iff P_j is a double point.

By following the fiber along the path γ, the points $\ell_{i_1} \cap \mathbb{C}_{\gamma(x_i', x_i'')(t)}, \ldots, \ell_{i_n} \cap \mathbb{C}_{\gamma(x_i', x_i'')(t)}$ move around inducing *half-twists* braids around the points with indices Δ_j, $|\Delta_j| > 1$ (see fig. 2).

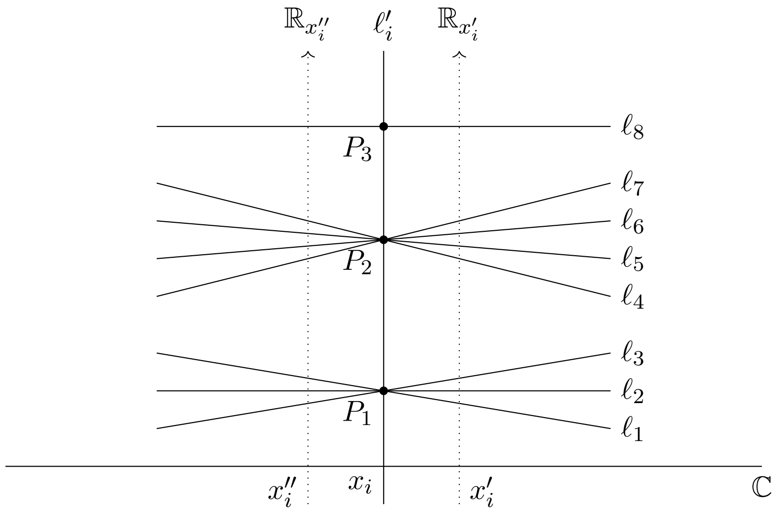

Figure 1. An example with $n = 8$, $\Delta_1 = \{1,2,3\}$, $\Delta_2 = \{4,5,6,7\}$ and $\Delta_3 = \{8\}$.

Then the generators $\alpha_k^{(x_i')}$, $k \in \Delta_j$, are transformed as

$$\gamma(x_i', x_i'')_*(\alpha_{a_j+h}^{(x_i')}) = \left(\prod_{p=a_j}^{a_j+m_j-2-h} \alpha_p^{(x_i'')}\right) \alpha_{a_j+m_j-1-h}^{(x_i'')} \left(\prod_{p=a_j}^{a_j+m_j-2-h} \alpha_p^{(x_i'')}\right)^{-1} \tag{2.11}$$

$(h = 0, \dots, m_j - 1)$.

By using (2.6), it follows that, when $k, k+1$ belong to the same block Δ_j,

$$\begin{aligned}
&\gamma(x_i', x_i'')_*(\tilde{\alpha}_{a_j+h,a_j+h+1}^{(x_i')}) = \\
&= [\gamma(x_i', x_i'')_*(\alpha_{a_j+h}^{(x_i')}\alpha_{a_j+h+1}^{(x_i')}(\alpha_{a_j+h}^{(x_i')})^{-1}(\alpha_{a_j+h+1}^{(x_i')})^{-1})] = \\
&= [P_h(\alpha_{a_j+m_j-1-h}^{(x_i'')}\alpha_{a_j+m_j-2-h}^{(x_i'')}(\alpha_{a_j+m_j-1-h}^{(x_i'')})^{-1}(\alpha_{a_j+m_j-2-h}^{(x_i'')})^{-1})P_h^{-1}]
\end{aligned}$$

where P_h is the expression in brackets in (2.11). This gives:

$$\gamma(x_i', x_i'')_*(\tilde{\alpha}_{a_j+h,a_j+h+1}^{(x_i')}) = -(\prod_{k=a_j+h+1}^{a_j+m_j-1} s_k^{(x')})\ \tilde{\alpha}_{a_j+m_j-2-h,a_j+m_j-1-h}^{(x_i'')} \tag{2.12}$$

which is equivalent to (2.10) in this case.

Let k be the last index of a block Δ_j and $k+1$ the first index of the following block Δ_{j+1}. To fix notation, suppose $\Delta_1 = \{1, \dots, k\},\ \Delta_2 = \{k+1, \dots, k+h\}$.

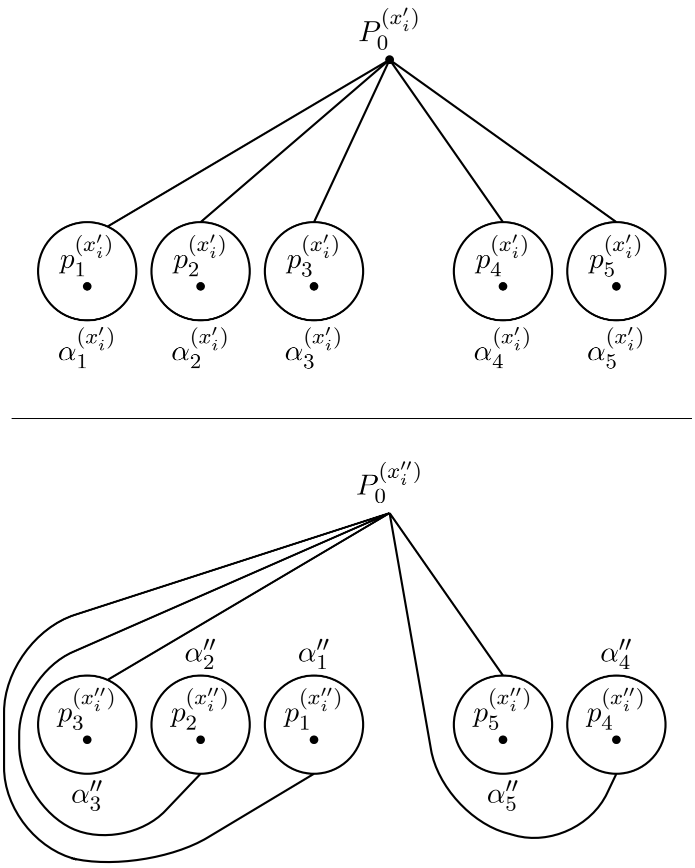

Figure 2. An example with $n = 5$, $\Delta_1 = \{1, 2, 3\}$ and $\Delta_2 = \{4, 5\}$. The figure on top represents $F_{x'_i}$, while the figure on the bottom represents $F_{x''_i}$.

We have

$$\gamma(x'_i, x''_i)_*(\tilde{\alpha}_{k,k+1}^{(x'_i)}) = [\alpha_1^{(x''_i)}, (\alpha_{k+1}^{(x''_i)} \dots \alpha_{k+h-1}^{(x''_i)})\alpha_{k+h}^{(x''_i)}(\alpha_{k+1}^{(x''_i)} \dots \alpha_{k+h-1}^{(x''_i)})^{-1}] =$$

(writing $s''_j = s_j^{(x''_i)}$ and $s'_j = s_j^{(x'_i)}$)

$$= (1 - s''_{k+h})[\alpha_1^{(x''_i)}] - (1 - s''_1)\{s''_{k+1} \dots s''_{k+h-1}[\alpha_{k+h}^{(x''_i)}] + (1 - s''_{k+h})([\alpha_{k+1}^{(x''_i)}] +$$
$$s''_{k+1}[\alpha_{k+2}^{(x''_i)}] + s''_{k+1}s''_{k+2}[\alpha_{k+3}^{(x''_i)}] + \cdots\cdots + s''_{k+1} \dots s''_{k+h-2}\ [\alpha_{k+h-1}^{(x''_i)}])\ \}\ =$$

(setting in general $\tilde{\alpha}^{(x)}_{i,j} = (1-s_j)[\alpha^{(x)}_i] - (1-s_i)[\alpha^{(x)}_j]$)

$$= \tilde{\alpha}^{(x''_i)}_{1,k+h} - (1-s''_1)(s''_{k+1}\dots s''_{k+h-2}\tilde{\alpha}^{(x''_i)}_{k+h-1,k+h} + s''_{k+1}\dots s''_{k+h-3}\tilde{\alpha}^{(x''_i)}_{k+h-2,k+h} + \dots + s''_{k+1}\tilde{\alpha}^{(x''_i)}_{k+2,k+h} + \tilde{\alpha}^{(x''_i)}_{k+1,k+h})$$

Now dividing by $(1-s^{(x')}_k)(1-s^{(x')}_{k+1}) = (1-s^{(x''_i)}_1)(1-s^{(x''_i)}_{k+h})$ we obtain

$$\gamma(x'_i,x''_i)_*(\alpha^{(x'_i)}_{k,k+1}) = \alpha^{(x''_i)}_{1,k+h} - s''_{k+1}\dots s''_{k+h-2}(1-s''_{k+h-1})\alpha^{(x''_i)}_{k+h-1,k+h} - s''_{k+1}\dots s''_{k+h-3}(1-s''_{k+h-2})\alpha^{(x''_i)}_{k+h-2,k+h} - \dots - (1-s''_{k+1})\alpha^{(x''_i)}_{k+1,k+h} =$$

(after writing, for $i<j$, $\alpha^{(x)}_{i,j} = \sum_{k=i}^{j-1}\alpha^{(x)}_{k,k+1}$ and grouping equal terms)

$$\begin{aligned} &= \alpha^{(x''_i)}_{1,2} + \dots + \alpha^{(x''_i)}_{k,k+1} + s''_{k+1}\alpha^{(x''_i)}_{k+1,k+2} + s''_{k+1}s''_{k+2}\alpha^{(x''_i)}_{k+2,k+3} + \dots + \\ &+ s''_{k+1}\dots s''_{k+h-1}\alpha^{(x''_i)}_{k+h-1,k+h} = \\ &= \alpha^{(x''_i)}_{1,2} + \dots + \alpha^{(x''_i)}_{k,k+1} + s'_{k+h}\alpha^{(x''_i)}_{k+1,k+2} + s'_{k+h}s'_{k+h-1}\alpha^{(x''_i)}_{k+2,k+3} + \dots + \\ &+ s'_{k+h}\dots s'_{k+2}\alpha^{(x''_i)}_{k+h-1,k+h} \end{aligned} \tag{2.13}$$

which is (2.10) in this case.

The general formula follows by induction on the number of the points $p \in \pi(\mathcal{S})$ contained in the interval $[x,x']$. QED

The local monodromy around a point $p \in \pi(\mathcal{S})$ is obtained by taking a point $x \in \mathbb{R}$ very close to p and transporting the fiber of π'' starting from x around a circle centered in p. If $x' \in \mathbb{R}$ is a point symmetric of x with respect to p, the local monodromy μ_x is the automorphism of $H_1(F_x;\mathcal{L}_x)$ which is obtained by composing $\gamma(x',x)\circ\gamma(x,x')$. We consider the "local permutation" $\sigma(x,x')$ as above, determined by a partition $\Delta_1,\Delta_2,\dots,\Delta_h$ of the set $\{1,\dots,n\}$, where each Δ is composed of consecutive numbers $a,a+1,\dots,a+r$ which are the indices (in the x-ordering) of the lines which intersect into a singular point P in the vertical line $\mathbb{C}_p$ (so the multiplicity of P as a singular point of $\mathcal{A}$ equals $\#\Delta+1$).

Theorem 2 (Local monodromy)**.** *With the previous notations, the local monodromy around p is given by*

$$\mu_x(\alpha^{(x)}_{i,i+1}) = \begin{cases} \Big(\prod_{j\in\Delta} s^{(x)}_j\Big)\alpha^{(x)}_{i,i+1} & (a) \\ \alpha^{(x)}_{i,i+1} + \sum_{k=a}^{i-1}(1-s^{(x)}_a\cdots s^{(x)}_k)\alpha^{(x)}_{k,k+1} + \\ \quad + \sum_{k=i+2}^{b} s^{(x)}_{i+1}\cdots s^{(x)}_{k-1}(1-s^{(x)}_k\cdots s^{(x)}_b)\alpha^{(x)}_{k-1,k} & (b) \end{cases} \tag{2.14}$$

where (a) occurs if i and $i+1$ belong to the same block Δ of $\sigma(x,x')$ and (b) occurs if i is the last element of the block $\{a, a+1, \ldots, i\}$ and $i+1$ is the first element of the next block $\{i+1, i+2, \ldots, b\}$.

Proof. Assume as above that $\Delta_1 = \{1, \ldots, k\}$, $\Delta_2 = \{k+1, \ldots, k+h\}$. By using $\mu_x(\alpha^{(x)}_{i,i+1}) = \gamma(x',x)_* \circ \gamma(x,x')_*$ we get:

- if $\{i, i+1\} \subset \Delta_1$ then (by (2.10) applied two times)

$$\mu_x(\alpha^{(x)}_{i,i+1}) = -\gamma(x',x)_*\Big[\Big(\prod_{j=i+1}^{k} s_j^{(x)}\Big)\alpha^{(x')}_{k-i,k-i+1}\Big] = \Big(\prod_{j=1}^{k} s_j^{(x)}\Big)\alpha^{(x)}_{i,i+1}$$

which is (a). Moreover

$$\begin{aligned}\mu_x(\alpha^{(x)}_{k,k+1}) = -\gamma(x',x)_*[\alpha^{(x')}_{1,2} + \cdots + \alpha^{(x')}_{k,k+1} + s^{(x)}_{k+h}\alpha^{(x')}_{k+1,k+2} + \cdots \\ + s^{(x)}_{k+h} \ldots s^{(x)}_{k+2}\alpha^{(x')}_{k+h-1,k+h}] =\end{aligned}$$

$$\begin{aligned}= \ -s_1^{(x)} \ldots s^{(x)}_{k-1}\alpha^{(x)}_{k-1,k} - \cdots - s_1^{(x)}\alpha^{(x)}_{1,2} + \gamma(x',x)_*(\alpha^{(x')}_{k,k+1}) - \\ -s^{(x)}_{k+1} \ldots s^{(x)}_{k+h}\Big(\alpha^{(x)}_{k+1,k+2} + \cdots + \alpha^{(x)}_{k+h-1,k+h}\Big)\end{aligned}$$

Since

$$\begin{aligned}\gamma(x',x)_*(\alpha^{(x')}_{k,k+1}) \ = \ \alpha^{(x)}_{1,2} + \cdots + \alpha^{(x)}_{k,k+1} + \\ + s^{(x)}_{k+1}\alpha^{(x)}_{k+1,k+2} + \cdots + s^{(x)}_{k+1} \ldots s^{(x)}_{k+h-1}\alpha^{(x)}_{k+h-1,k+h}\end{aligned}$$

case (b) follows.

QED

Let $p \in \pi(\mathcal{S})$ and let $P_1, \ldots, P_h$ be the singular points of $\mathcal{A}$ in the vertical line $\mathbb{C}_p$, having multiplicity $\mu_1, \ldots, \mu_h$ respectively ($\mu_1 + \cdots + \mu_h = n+h$). Then each block Δ corresponds to some singular point P_i and from (2.14) $\prod_{j\in\Delta} s_j^{(x)} = \prod_{P_i\in\ell} s(\ell)$ is an eigenvalue, of multiplicity $\mu_i - 2 = m_i - 1$ ($|\Delta| = \mu_i - 1 = m_i$).

From (2.14), (b), it follows easily that 1 is an eigenvalue, of multiplicity at least $h-1$, with equality iff each $\prod_{P_i\in\ell} s(\ell) \neq 1$. In fact

Corollary 1. *The local monodromy μ_x has characteristic polynomial*

$$p_x(\lambda) = (\lambda - 1)^{h-1}\prod_{i=1}^{h}(\ \lambda - \prod_{P_i\in\ell} s(\ell)\ )^{m_i-2} \tag{2.15}$$

Assuming all $s_i \neq 1$, μ_x is diagonalizable iff all eigenvalues of the form $\prod_{P_i\in\ell} s(\ell)$ are different from 1.

Proof. From (2.14), the $(n-1)\times(n-1)$ matrix M_1 of $\mu_x - id$ has $h-1$ zero rows in correspondence with the generators $\alpha^{(x)}_{k,k+1}$ for k the last element of a block, $k+1$ the first element of the next block. If $\prod_{P_i\in\ell} s(\ell)\neq 1$ for each singular point P_i, the rank of M_1 is $n-h$. If some $\prod_{P_i\in\ell} s(\ell)=1$, the algebraic multiplicity of 1 grows by $m_i - 2 = |\Delta_i| - 1$ but the rank does not diminish by the same quantity, since some coefficient $(1-s^{(x)}_a\dots s^{(x)}_k)\neq 0,\ a\leq k<i$. $\boxed{QED}$

Now we consider the global monodromy $\mu\colon \pi_1(B,x_0)\to \mathrm{Aut}(H_1(F_0;\mathcal{L}_s))$. For each point $x_k\in\pi(\mathcal{S})$ we can use the well ordered set of generators γ_k of $\pi_1(B,x_0)$ which is defined in (2.9). (An explicit formula for the global monodromy is given in [23] by using paths of the form $\gamma(x_0,x''_k)\gamma(x''_k,x_0)$, but we don't need it here). By construction one has:

Theorem 3 (Global monodromy)**.** *The global monodromy μ is determined by the maps*

$$\mu([\gamma_p]) = \gamma(x_0,x'_k)^{-1}_*\mu_{x'_k}\gamma(x_0,x'_k)_* \tag{2.16}$$

where $\gamma(x_0,x'_k)_*$ is the parallel transport in (2.10) *and $\mu_{x'_k}$ is the local monodromy in* (2.14)*.* $\boxed{QED}$

We now come back to the computation of $H_0(B;H_1(F_0;\mathcal{L}_s))$ in (2.2). The space B deformation retracts onto a wedge of 1-spheres, actually onto the union $\cup_{p\in\pi(S)}\gamma_p$, where the unique 0-cell is x_0. By remark 1 we obtain

Theorem 4. *The vector space $H_0(B;H_1(F_0;\mathcal{L}_s))$ is the cokernel of the map*

$$\partial_1\colon \bigoplus_{p\in\pi(\mathcal{S})} H_1(F_0;\mathcal{L}_s)[\gamma_p]\to H_1(F_0;\mathcal{L}_s)[x_0] \tag{2.17}$$

defined as

$$\partial_1(\alpha[\gamma_p]) = \Big(t_p\cdot\mu([\gamma_p])(\alpha)-\alpha\Big)[x_0] \tag{2.18}$$

where we set here t_p as the parameter associated with the vertical line $\mathbb{C}_p\in\mathcal{A}$. $\boxed{QED}$

Let M_p be the matrix of $\mu([\gamma_p])$ in the given basis. Then the matrix B of ∂_1 consists of m square blocks $B_p = t_pM_p - \mathrm{Id}$, one for each $p\in\pi(\mathcal{S})$, of order $n-1$, so $B=[B_1,\dots,B_m]$. It follows that the block B_p has rank lower than $n-1$ iff t_p coincides with the inverse of an eigenvalue of $\mu([\gamma_p])$. Therefore either $t_p=1$ or $t_p\prod_{P_i\in\ell}s_\ell = 1$ for some singular point $P_i\in\mathbb{C}_p$ of order at least 3 (see (2.15)). We obtain

Theorem 5. *The essential part of the characteristic variety $V(\mathcal{A})$ coincides with the set of values $(s,t)\in(\mathbb{C}^*)^{n+m}$ such that the transpose of the monodromy* *operators $\mu([\gamma_p])(s)$, $p \in \pi(\mathcal{S})$, have a common eigenvector relative to eigenvalues t_p^{-1}, $p \in \pi(\mathcal{S})$.*

Proof. In fact, a linear relation among the rows of the boundary matrix B is given by a vector v such that ${}^t v B_p = {}^t v(t_p M_p - \mathrm{Id}) = 0$, $p \in \pi(\mathcal{S})$. $\boxed{QED}$

3 The general case

Let $\mathcal{A}'$ be a line arrangement not necessarily fibered. Therefore, some (possibly all) vertical lines containing the singular points do not belong to $\mathcal{A}'$. However, we can embed $\mathcal{A}'$ into a fibered arrangement $\mathcal{A}$ of the type in section 2, where $\mathcal{A} \setminus \mathcal{A}'$ is given by the vertical lines $\ell'_{k_1}, \ldots, \ell'_{k_h}$ $(k_1 < \cdots < k_h)$. Let $\mathcal{L}'_{s,t}$ be the local system on $\mathcal{M}(\mathcal{A}')$ with parameters $s_i,\ i = 1, \ldots, n,$ and $t_j,\ j = 1, \ldots, m, j \neq k_1, \ldots, k_h$. Let also $\mathcal{L}^1_{s,t}$ be the local system on $\mathcal{M}(\mathcal{A})$ with parameters s_i, t_j as in part 2 but such that $t_{k_1} = \cdots = t_{k_h} = 1$. Notice that $\mathcal{L}^1_{s,t} = i^*(\mathcal{L}'_{s,t})$, where $i : \mathcal{M}(\mathcal{A}) \to \mathcal{M}(\mathcal{A}')$ is the inclusion.

We also consider the $1-$dimensional arrangement $\mathcal{A}''$ on $\ell'_{k_1} \cup \cdots \cup \ell'_{k_h} \cong \mathbb{C} \sqcup \cdots \sqcup \mathbb{C}$ (h copies) given by $\ell'_{k_j} \cap \ell_i,\ i = 1, \ldots, n,\ j = 1, \ldots, h$. We have an induced local system $\mathcal{L}''_s = j^*(\mathcal{L}'_{s,t})$ on the complement $\mathcal{M}(\mathcal{A}'')$, where $j : \mathcal{M}(\mathcal{A}'') \to \mathcal{M}(\mathcal{A}')$ is the inclusion.

For the triple $(\mathcal{L}^1_{s,t}, \mathcal{L}'_{s,t}, \mathcal{L}''_s)$ we have ([5])

Theorem 6. *There are long exact sequences*

$$\begin{aligned} \to H^q(\mathcal{M}(\mathcal{A}'); \mathcal{L}'_{s,t}) \to H^q(\mathcal{M}(\mathcal{A}); \mathcal{L}^1_{s,t}) \to & H^{q-1}(\mathcal{M}(\mathcal{A}''); \mathcal{L}''_s) \to \\ & \to H^{q+1}(\mathcal{M}(\mathcal{A}'); \mathcal{L}'_{s,t}) \to \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \to H_{q+1}(\mathcal{M}(\mathcal{A}'); \mathcal{L}'_{s,t}) \to H_{q-1}(\mathcal{M}(\mathcal{A}''); \mathcal{L}''_s) \to & H_q(\mathcal{M}(\mathcal{A}); \mathcal{L}^1_{s,t}) \to \\ & \to H_q(\mathcal{M}(\mathcal{A}'); \mathcal{L}'_{s,t}) \to \end{aligned} \tag{3.2}$$

$\boxed{QED}$

Sequence (3.1) is shown in [5, thm.4]; sequence (3.2) is similar. As a corollary we have ([5, Cor.5])

Corollary 2. *If the restriction of $\mathcal{L}''_s$ to each vertical line ℓ'_{k_j} is nontrivial then*

$$H^1(\mathcal{M}(\mathcal{A}'); \mathcal{L}'_{s,t}) \cong H^1(\mathcal{M}(\mathcal{A}); \mathcal{L}^1_{s,t})$$

and

$$H_1(\mathcal{M}(\mathcal{A}'); \mathcal{L}'_{s,t}) \cong H_1(\mathcal{M}(\mathcal{A}); \mathcal{L}^1_{s,t})$$

Proof. In fact, in this case one has $H^0(\mathcal{M}(\mathcal{A}''),\mathcal{L}''_s) = H_0(\mathcal{M}(\mathcal{A}''),\mathcal{L}''_s) = 0$ and the isomorphisms follow from sequences 3.1 and 3.2. $\boxed{QED}$

Notice that the restriction of $\mathcal{L}''_s$ to the line ℓ'_{k_j} is nontrivial iff there exists a singular point $P \in \ell'_{k_j}$ such that $\prod_{P\in\ell} s(\ell) \neq 1$.

Under this assumption, we can use the complex constructed in section 2 to compute $H_1(\mathcal{M}(\mathcal{A}');\mathcal{L}'_{s,t})$, by setting the parameters $t_{k_j} = 1,\ j = 1,\dots,h$.

Theorem 7. *Assuming that $\mathcal{L}''_s$ is nontrivial on each line ℓ'_{k_j}, one has that $H_1(\mathcal{M}(\mathcal{A}');\mathcal{L}'_{s,t})$ is the cokernel of the map (2.17), where δ_1 is given as in (2.18) setting $t_p = 1$ for $p = k_1,\dots,k_h$.*

In particular, assume that the coordinate system is generic: then two singular points never belong to the same vertical line, and no vertical line is contained in $\mathcal{A}'$. In this case all $t_p = 1$ and $\mathcal{L}'_{s,t}$ depends only on the parameters s_i, so we set in this case $\mathcal{L}'_s := \mathcal{L}'_{s,t}$. In this situation, unless the arrangement $\mathcal{A}'$ is a pencil, each vertical line ℓ'_{k_j}, containing a singular point P of $\mathcal{A}'$, intersects all lines of $\mathcal{A}'$ which do not contain P in a double point. Therefore, if all parameters s_i are different from 1 then the local system $\mathcal{L}''_s$ is nontrivial on each vertical line ℓ'_{k_j}.

Corollary 3. *Under the previous genericity conditions, $H_1(\mathcal{M}(\mathcal{A}');\mathcal{L}'_s)$ is the cokernel of the map (2.17), where δ_1 is given as in (2.18) setting $t_p = 1$ for all p.*

The global components of the characteristic variety are given by those parameters s such that the transposes of the monodromy operators fix a common vector ($\neq 0$).

The second statement follows from theorem 5, where all t_i are 1.

4 Milnor fiber

We now consider the (co-)homology of the Milnor fiber (see also [4], [8], [18],[21], [22],[28], [29], [31]). We recall some notations.

Let $\mathcal{A} = \{H_1,\dots,H_n\}$ be an affine hyperplane arrangement in $\mathbb{C}^m$ with coordinates $z_1,\dots,z_m$ and, for every $1 \le i \le n$ let α_i be a linear polynomial such that $H_i = \alpha_i^{-1}(0)$. The *cone* $c\mathcal{A}$ of $\mathcal{A}$ is a central arrangement in $\mathbb{C}^{m+1}$ with coordinates $z_0,\dots,z_m$ given by $\{\widetilde{H_0},\widetilde{H_1},\dots,\widetilde{H_n}\}$ where $\widetilde{H_0}$ is the coordinate hyperplane $z_0 = 0$ and, for every $1 \le i \le n$, $\widetilde{H_i}$ is the zero locus of the homogenization of α_i with respect to z_0.

Now let $\widetilde{\mathcal{A}} = \{\widetilde{H}_0,\dots,\widetilde{H}_n\}$ be a central arrangement in $\mathbb{C}^{m+1}$ and choose coordinates $z_0,\dots,z_m$ such that $\widetilde{H}_0 = \{z_0 = 0\}$; moreover, for every $1 \le i \le n$, let $\widetilde{\alpha}_i(z_0, \ldots, z_n)$ be such that $\widetilde{H_i} = \widetilde{\alpha_i}^{-1}(0)$. The *deconing* of $\widetilde{\mathcal{A}}$ is the arrangement $d\widetilde{\mathcal{A}}$ in $\mathbb{C}^m$ given by $\{H_1, \ldots, H_n\}$ where, if we set for every $1 \leq i \leq n$, $\alpha_i(z_1, \ldots, z_n) = \widetilde{\alpha}_i(1, z_1, \ldots, z_n)$, $H_i = \alpha_i^{-1}(0)$. One sees easily that $\mathcal{M}(c\mathcal{A}) = \mathcal{M}(\mathcal{A}) \times \mathbb{C}^*$ (and conversely $\mathcal{M}(\widetilde{\mathcal{A}}) = \mathcal{M}(d\widetilde{\mathcal{A}}) \times \mathbb{C}^*$).

In dimension 3, let $Q : \mathbb{C}^3 \to \mathbb{C}$ be a homogeneous polynomial (of degree $n+1$) which defines the arrangement $\widetilde{\mathcal{A}}$. Then Q gives a fibration

$$Q_{|\mathcal{M}(\tilde{\mathcal{A}})} : \mathcal{M}(\tilde{\mathcal{A}}) \to \mathbb{C}^*$$

with *Milnor fibre*

$$\mathbf{F} = Q^{-1}(1)$$

and *geometric monodromy*

$$\pi_1(\mathbb{C}^*, 1) \to Aut(F)$$

induced by $x \to e^{\frac{2\pi i}{n+1}} \cdot x$.

Let A be any unitary commutative ring and

$$R := A[s, s^{-1}].$$

Consider the abelian representation

$$\pi_1(\mathcal{M}(\widetilde{\mathcal{A}})) \to H_1(\mathcal{M}(\widetilde{\mathcal{A}}); \mathbb{Z}) \to Aut(R) : \ \beta_j \to s\cdot$$

taking a generator β_j into s-multiplication. Let R_s be the ring R endowed with this $\pi_1(\mathcal{M}(\widetilde{\mathcal{A}}))-$module structure. Therefore R_s induces a local system of coefficients over $\mathcal{M}(\widetilde{\mathcal{A}})$.

The following proposition is well-known, coming from the homotopy equivalence between F and the infinite cyclic covering determined by $\pi_1(\mathcal{M}(\widetilde{\mathcal{A}})) \to \pi_1(\mathbb{C}_*)$, with deck transformations generated by the monodromy.

Proposition 1. *One has an R-module isomorphism*

$$H_*(\mathcal{M}(\widetilde{\mathcal{A}}), R_s) \cong H_*(F, A)$$

where $s-$multiplication on the left corresponds to the monodromy action on the right.

In particular for $R = \mathbb{Q}[s, s^{-1}]$, which is a PID, one has

$$H_*(\mathcal{M}(\widetilde{\mathcal{A}}), \mathbb{Q}[s^{\pm 1}]) \cong H_*(F, \mathbb{Q}).$$

Since the monodromy operator has order dividing $n+1$ then $H_*(\mathcal{M}(\widetilde{\mathcal{A}}); R_s)$ decomposes into cyclic modules either isomorphic to R or to $\frac{R}{(\varphi_d)}$, where φ_d is a cyclotomic polynomial, with $d|n+1$.

It follows from the associated spectral sequence, remark 1 and from the fact that the monodromy acts trivially in dimension zero that

$$n+1 = dim(H_1(\mathcal{M}(\widetilde{\mathcal{A}});\mathbb{Q})) = 1 + dim\frac{H_1(F;\mathbb{Q})}{(\mu-1)}$$

where on the right one has the coinvariants w.r.t. the monodromy action. Therefore it always holds

$$b_1(F) \geq \ n.$$

Precisely, one has

$$b_1(F) = n \quad \Leftrightarrow \quad \mu = id.$$

In [24] we called *a-monodromic* an arrangement $\widetilde{\mathcal{A}}$ having trivial monodromy on $H_1(F;\mathbb{Q})$, and noticed that $\widetilde{\mathcal{A}}$ is a-monodromic iff $H_1(F;\mathbb{Q}) \cong \mathbb{Q}^n$ (equivalently: $H_1(\mathcal{M}(\widetilde{\mathcal{A}}); R_s) \cong \left(\frac{R}{(s-1)}\right)^n$ where $R = \mathbb{Q}[s, s^{-1}]$).

We continue to indicate by R_s the local system restricted to the affine part $\mathcal{M}(\mathcal{A})$, $\mathcal{A} = d\widetilde{\mathcal{A}}$. When $A = \mathbb{C}$, it coincides with the particular local system $\mathcal{L}'_s$ defined in the previous section, where all parameters s_i equal the same $s \in \mathbb{C}^*$. In [24] we also defined a-monodromicity of the affine part $\mathcal{A}$: we call $\mathcal{A}$ *a-monodromic* if $H_1(\mathcal{M}(\mathcal{A}); R_s) \cong \left(\frac{R}{(s-1)}\right)^{n-1}$.

By Kunneth formula (with $A = \mathbb{Z}$ or A a field as $\mathbb{Q}$ or $\mathbb{C}$) it follows

$$H_1(\mathcal{M}(\widetilde{\mathcal{A}}); R_s) \cong H_1(\mathcal{M}(\mathcal{A}); R_s) \otimes \frac{R}{(s^{n+1}-1)} \oplus \frac{R}{(s-1)}. \tag{4.1}$$

Therefore, if $\mathcal{A}$ has trivial monodromy then $\widetilde{\mathcal{A}}$ does (the converse is not true in general (see [24])).

In [24] we stated the conjectures

Conjecture 1. *let Γ be the graph with vertex set $\mathcal{A}$ and edge-set all pairs $\{\ell_i, \ell_j\}$ such that $\ell_i \cap \ell_j$ is a double point. Then if Γ is connected then $\mathcal{A}$ is a-monodromic.*

Conjecture 2. *let Γ be as before. Then if Γ is connected then $\widetilde{\mathcal{A}}$ is a-monodromic.*

By formula (4.1) conjecture 1 implies conjecture 2.

5 Some computations

We now come back to theorem 7 and its corollary 3 to compute the local system homology. We consider the same genericity conditions, and we consider here the local system R_s above (equivalently, the local system $\mathcal{L}'_s$ with all parameters $s_i = s,\ i = 1, \ldots, n$).

We consider the boundary operator given in (2.18), analyzing each of its blocks (of order $n-1$) relative to some double point of $\mathcal{A}'$. According to equation 2.18 and theorem 7 such a block is constructed as

$$\mu([\gamma_p]) - Id$$

where $\mu([\gamma_p])$ is the matrix of the monodromy around the projection p of the double point. As explained in formula (2.16), one has $\mu([\gamma_p]) = \gamma(x_0, x'_p)_*^{-1} \mu_{x'_p} \gamma(x_0, x'_p)_*$, where $\mu_{x'_p}$ is the local monodromy. Therefore

$$\mu([\gamma_p]) - Id = \gamma(x_0, x'_p)_*^{-1} (\mu_{x'_p} - Id) \gamma(x_0, x'_p)_* \tag{5.1}$$

By the explicit form (2.14) of the local monodromy, it follows that $rk(\mu_{x'_p} - Id) = 1$; precisely

$$\mu_{x'_p} - Id \ = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & s(1-s) & s^2-1 & 1-s & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

where the non-zero position k in the $(k-1)$-th, k-th, $(k+1)$-th columns corresponds to the double point formed by the $k, k+1$-th lines (in local ordering).

It follows that $rk(\mu([\gamma_p]) - Id) = 1$. More precisely one obtains.

Theorem 8. *The column-echelon form of $\mu([\gamma_p]) - Id$ has a unique non-zero column C^p; if $p = \pi(P)$ where $P = \ell_i \cap \ell_j,\ i < j$, then*

$$\begin{array}{lcll} C^p_k & = & 0 & \textit{if } k \textit{ does not belong to the semi-open interval } [i, j) \\ C^p_{i+h} & = & (s-1)\ s^{\mu(h)}, & h = 0, \ldots j-i-1. \end{array} \tag{5.2}$$

Here we set $\mu(h) = \#\{a :\ i \le a \le i+h,\ \ell_a \text{ separates } P \text{ from } x_0\}$.

Proof. The fact that there is a unique column derives from $rk(\mu([\gamma_p]) - Id) = 1$.

Let $A = \mu_{x'_p} - Id$, $P = \gamma(x_0, x'_p)_*$. Then the column-echelon form of $\mu([\gamma_p]) - Id = P^{-1}AP$ is the same as the column-echelon form of $P^{-1}A$. This is a matrix whose only non-zero columns are the $(k-1)$-th, k-th, $(k+1)$-th, where k is the index of the only non-zero row in A. Such three columns are $(s-1)$-multiples of the k-th column of P^{-1}. Then (5.2) follows directly from formula (2.10). $\boxed{QED}$

We now generalize some results in [24]. First, we say that a connected graph Γ with vertices $\{1,\ldots,n\}$ is *inductive* if:

- $n=2$, or,
- there exists $i\in\{1,\ldots,n\}$ such that $\{i,i+1\}$ is an edge of Γ and if we define $\varphi_i:\{1,\ldots,n\}\to\{1,\ldots,n-1\}$ to be the increasing surjective function with $\varphi_i(i)=\varphi_i(i+1)$, and we define the graph Γ' having vertices $\{1,\ldots,n-1\}$ and edges $\{\varphi_i(a),\varphi_i(b)\}$ for each $\{a,b\}$ edge of Γ distinct from $\{i,i+1\}$, then Γ' is inductive. In other words, the graph obtained from Γ by identifying two consecutive vertices which are connected by an edge and by updating the vertex labels accordingly is inductive.

See Figures 3 and 4 for an example and a proof that a graph is inductive. We have this lemma:

Lemma 3. *Let Γ be an inductive graph with vertices $\{1,\ldots,n\}$, let R be a PID, and let M be a matrix with entries in R with $n-1$ rows and a column C^e for each edge $e=\{a,b\}, a<b$, of Γ, satisfying:*

$$\begin{array}{lcll} C_k^{\{a,b\}} & = & 0 & \text{if } k \text{ does not belong to the semi-open interval } [a,b) \\ C_{a+h}^{\{a,b\}} & \in & R^*, & h=0,\ldots b-a-1. \end{array} \tag{5.3}$$

Then, the Smith form of M is the identity.

Proof. We prove this by induction on n. For $n=2$ this is obvious, as M is a 1×1 matrix with an invertible element as its only entry. Assuming $n>2$, fix i,Γ' as in the definition of inductive graph. Since $C^{\{i,i+1\}}$ is the elementary basis element e_i multiplied by an element of R^*, it is easy to see that the statement is equivalent to the Smith form of the matrix M' being the identity, where M' is the matrix obtained from M by removing the column $C^{\{i,i+1\}}$ and the i-th row. Notice that M' satisfies the assumption with Γ', hence the statement follows by inductive hypothesis. $\boxed{QED}$

From Lemma 3 we deduce:

Theorem 9. *Let $\mathcal{A}$ be an affine arrangement, and number its lines $\{\ell_1,\ldots,\ell_n\}$ according to some coordinate system. Let Γ be the* graph of double points *of $\mathcal{A}$, i.e. the graph with vertex set $\{1,\ldots,n\}$ and edge-set all pairs $\{i,j\}$ such that $\ell_i\cap\ell_j$ is a double point of $\mathcal{A}$. Suppose Γ is inductive. Then $\mathcal{A}$ is a-monodromic.*

Proof. We consider the submatrix of the boundary matrix ∂_1 given by all the blocks relative to double points. By Theorem (8) ∂_1 can be column-reduced to a matrix $(s-1)M_s$, where the columns of M_s are given by (5.2) (one column for each edge of Γ). It is sufficient to show that the matrix M_s has Smith form the identity matrix, and this follows immediately from Lemma 3. $\boxed{QED}$

Recall from [24, Definition 5.3]: a connected graph Γ with vertices $\{1,\ldots,n\}$ is called *good* if there exists a permutation $i_1,\ldots,i_n$ of $1,\ldots,n$ such that, for each $k\in\{1,\ldots,n\}$, i_k is either the maximum or the minimum of $\{i_k,i_{k+1},\ldots,i_n\}$, and the full subgraph of Γ spanned by $\{i_k,i_{k+1},\ldots,i_n\}$ is connected. We say that $i_1,\ldots,i_n$ is a *good collapsing sequence* for Γ.

Proposition 2. *Let $n\geq 2$, and let Γ be a good graph with vertices $\{1,\ldots,n\}$. Then, Γ is inductive.*

Proof. We aim to prove the statement by induction on n. If $n=2$, Γ is obviously inductive, so we consider $n>2$. Let $i_1,\ldots,i_n$ be a collapsing sequence for Γ, and define $i:=\min\{i_n,i_{n-1}\}$. Notice that $\{i_n,i_{n-1}\}=\{i,i+1\}$, and thus $\{i,i+1\}$ is an edge of Γ. Moreover, if we define φ_i and Γ' as in the definition of inductive graph, we have that $\varphi_i(i_1),\ldots,\varphi_i(i_{n-1})$ is a good collapsing sequence for Γ', which is therefore a good graph. The inductive hypothesis yields that Γ' is an inductive graph, and thus Γ is also inductive. $\boxed{QED}$

Finally, we give an example that shows that Theorem 9 is indeed a generalization of [24, Theorem 3].

Proposition 3. *There exists an affine arrangement $\mathcal{A}$ such that:*

- *for each coordinate system, the graph of the double points of $\mathcal{A}$ is not good, and*
- *there exists a coordinate system such that graph of the double points of $\mathcal{A}$ is inductive.*

Proof. Let $\mathcal{A}$ be the arrangement from Figure 3, and let Γ be its graph of double points with the coordinate system from that figure. Figure 4 illustrates why Γ is inductive. Moreover, since the full subgraphs of Γ spanned by $\{1,2,3,4,5\}$, $\{2,3,4,5,6\}$ and $\{2,3,4,5\}$ are all disconnected by both 2 and 5, Γ is not good. Finally, notice that any coordinate system yields a graph of double points that is either Γ or a graph that is not inductive, so either way that graph is not good. $\boxed{QED}$

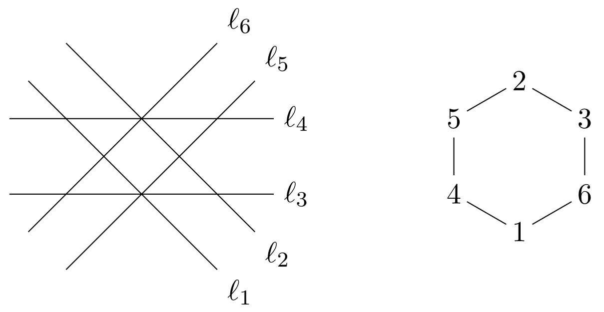

Figure 3. An affine arrangement satisfying the statement of Proposition 3, and its graph of double points.

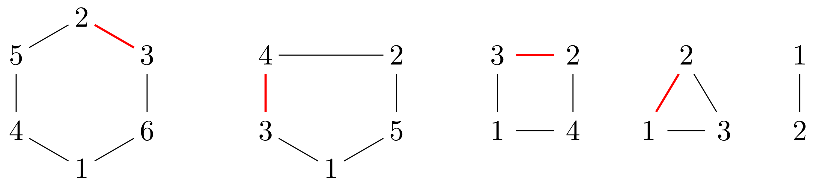

Figure 4. Proof that the graph from Figure 3 is inductive.

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