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Weak acceleration for interfacial motions

Syota Koide
Department of Science, National Institute of Technology, Matsue College
koide.syota@matsue-ct.ac.jp

Seiro Omata[i]
Osaka Central Advanced Mathematical Institute, Osaka Metropolitan University
seiro.omata@gmail.com

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Abstract. We provide two formulations of the acceleration for the motion of the interfaces in the framework of geometric measure theory. The first formulation is considered using varifolds, which are generalised surfaces seen as measures. The second formulation is considered using Caccioppoli sets, which are known as sets of finite perimeter. Both formulations arise from the time derivative of an integral quantity on the hypersurface associated with a weighted momentum. We further prove that the two notions of weak acceleration coincide under suitable assumptions. This result establishes a weak formulation of acceleration relevant to the study of weak solutions of hyperbolic mean curvature flow.

Keywords: interface acceleration, weak formulation, varifolds, Caccioppoli sets, hyperbolic mean curvature flow

MSC 2020 classification: primary 49Q20, secondary 53E10

Introduction

In this paper, we introduce two weak formulations of the acceleration for interface evolution, and show that they coincide under suitable assumptions. Let $\{\Gamma_t\}_{t\geq 0}$ be a family of hypersurfaces in $\mathbb{R}^N$ with time parameter t; we call $\{\Gamma_t\}_{t\geq 0}$ a hyperbolic mean curvature flow (mean curvature accelerated flow) if for any $t > 0$

$$D_t v = \kappa \quad \text{on } \Gamma_t, \tag{1}$$

where $D_t v = D_t v(x,t)$ denotes the normal time derivative of the normal velocity v at $x \in \Gamma_t$, and $\kappa = \kappa(x,t)$ is the mean curvature at $x \in \Gamma_t$. Here, we set the suitable initial conditions given by Γ_0 and $v(\cdot,0) = v_0(\cdot)$ on Γ_0. This equation models phenomena such as the melting-crystallising wave on a helium crystal [6] and the motion of soap bubbles [13]. He et al. [8] proved the local existence and uniqueness of the solution for the vector-valued version of (1). Ginder and Svadlenka [7] proposed a hyperbolic BMO algorithm for multi-phase and

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volume-conservation problems. We now turn to weak solutions for the equation (1). LeFloch and Smoczyk [10] derived a hyperbolic mean curvature flow with a certain conservative structure by a variational principle, and constructed a weak solution by a one-dimensional graph. Belletini et al. [3] constructed a weak solution for the variant of the equation $D_t v = (1 - v^2)\kappa$ by Lipschitz closed curves. Bellettini et al. [4] also introduced Lorentzian varifolds for the study of closed relativistic strings, providing an example of the application of geometric measure theory to hyperbolic geometric evolution problems. However, weak notions of acceleration for varifolds and Caccioppoli sets have not yet been established. Weak solutions for mean curvature flow equations $v = \kappa$ have been developed by Brakke using varifolds [5], by Almgren et al. using currents [1], Luckhaus and Sturzenhecker using Caccioppoli sets [9], where the normal velocity is formulated via integral identities. By analogy, it is natural to expect that a weak formulation of acceleration in terms of integral identities plays a key role in the study of hyperbolic mean curvature flow. On the other hand, the variational formula for acceleration is provided by the author in his dissertation [15], which would motivate the weak formulation of the acceleration for varifolds. In light of this situation, we consider the weak formulation of the acceleration for both varifolds (denoted by a_{var}) and Caccioppoli sets (denoted by a_{cac}). The main result of this paper is as follows.

Theorem. *Let $\{E_t\}_{t\in[0,T)}$ be a family of Caccioppoli sets in $\mathbb{R}^N$ with suitable assumptions (see Theorem 1 below). Then, the two notions of the acceleration coincide, i.e., $a_{\mathrm{cac}}(x,t) = a_{\mathrm{var}}(x,t)$ for $(\mathscr{H}^{N-1} \times dt)$-almost every (x,t).*

This result establishes the consistency of the two weak formulations of acceleration, which is a key step toward the study of weak solutions of hyperbolic mean curvature flow.

The organization of this paper is as follows. At the end of this introduction, we briefly review the key concepts used throughout the paper, namely moving hypersurfaces, varifolds, and Caccioppoli sets. In Section 1, we present two weak formulations of acceleration. Motivated by the classical setting, we define weak acceleration for families of time-evolving varifolds and Caccioppoli sets. In Section 2, we show that the two notions of acceleration coincide under suitable conditions.

Notation and some preliminaries

(i) Moving hypersurface

We prepare some notations about the theory of moving hypersurfaces from [14, 15]. Let $(\Gamma_t)_{t\geq 0}$ be a time dependent family of nonempty oriented $(N-1)$-

dimensional hypersurfaces (possibly with boundary) in $\mathbb{R}^N$. We define *moving hypersurfaces* in $\mathbb{R}^N$, denoted $\mathcal{M}$ by $\bigcup_{t\geq 0}(\Gamma_t \times \{t\})$ (this is usually called sapce-time track) that is, $(x_0, t_0) \in \mathcal{M}$ means that $x_0 \in \Gamma_{t_0}$, and $\mathcal{M}$ is a subset in $\mathbb{R}^{N+1}$. For each $(x,t) \in \mathcal{M}$, we define $\kappa(x,t)$, $\boldsymbol{n}(x,t)$ as the mean curvature, unit normal vector at $x \in \Gamma_t$ respectively. Hereafter, we assume that $\mathcal{M}$ is a C^1-class N-dimensional hypersurface in $\mathbb{R}^{N+1}$ and $\boldsymbol{n} \in C^1(\mathcal{M}, \mathbb{R}^N)$. For each $(x,t) \in \mathcal{M}$, we define *the normal velocity* $v(x,t)$ by $\boldsymbol{y}'(t) \cdot \boldsymbol{n}(x,t)$. Here, $\boldsymbol{y}$ is a C^1 *trajectory* on $\mathcal{M}$ if $\boldsymbol{y} \in C^1(\mathcal{I}_0, \mathbb{R}^N)$ with $\boldsymbol{y}(t) \in \Gamma_t$ for $t \in \mathcal{I}_0$. v does not depend on trajectory $\boldsymbol{y}$ (see [14, Theorem 5.5]). It is known that for any $(x_0, t_0) \in \mathcal{M}$, there exists a unique C^1 trajectory $\boldsymbol{y}$ such that $\boldsymbol{y}'(t) = v(\boldsymbol{y}(t), t)\boldsymbol{n}(\boldsymbol{y}(t), t)$ (see [14, Proposition 5.8]). Such $\boldsymbol{y}$ is called *the normal trajectory* on $\mathcal{M}$ through (x_0, t_0). We note that, in the smooth setting, the normal velocity can also be characterized in terms of the signed distance function (see [2, Theorem 7 and 8, pp. 18–20]).

Next, we introduce the notion of *the normal time derivative*. This answers the question of what is the time derivative of the functions $f(x,t)$ on $\mathcal{M}$. Let $f \in C^1(\mathcal{M}, \mathbb{R}^k)$, $k = 1, \ldots, N+1$. For $(x_0, t_0) \in \mathcal{M}$, and the normal trajectory $\boldsymbol{y}$ on $\mathcal{M}$ through (x_0, t_0),

$$D_t f(x_0, t_0) := \left. \frac{d}{dt} f(\boldsymbol{y}(t), t) \right|_{t=t_0},$$

$D_t f$ is called *the normal time derivative* of f on $\mathcal{M}$. Remark that $D_t f$ does not depend on the trajectory because the normal trajectory is unique, and even if a function does not depend on the time variable, the normal time derivative may be not zero. We also have for any $f \in C^1(\mathcal{Q})$ where $\mathcal{Q}$ is an open neighbourhood of $\mathcal{M}$ in $\mathbb{R}^{N+1}$, that the normal time derivative can be expressed as $D_t f = f_t + v \nabla f \cdot \boldsymbol{n}$.

We recall the following *transport identities* from [14, Theorems 6.1 and 6.4], stated there in a more general setting but restricted here to the case where $\Gamma_t = \partial \Omega_t$ for a bounded open set Ω_t.

$$\frac{d}{dt} \int_{\Gamma_t} f \, d\mathscr{H}^{N-1} = \int_{\Gamma_t} (D_t f - f\kappa v) \, d\mathscr{H}^{N-1} \quad (f \in C^1(\mathcal{M})). \tag{2}$$

$$\frac{d}{dt} \int_{\Omega_t} f \, d\mathscr{L}^N = \int_{\Omega_t} f_t \, \mathscr{L}^N - \int_{\Gamma_t} f v \, d\mathscr{H}^{N-1} \quad (f \in C^1(\overline{Q})) \tag{3}$$

where $Q := \bigcup_{t\geq 0}(\Omega_t \times \{t\})$. The identity (2) provides the variational structure that essentially underlies the Brakke's mean curvature flow (cf. the identity (4), see [5]).

(ii) Varifolds

Here we gather some properties of varifolds from [16]. Let $G_{N-1}(\mathbb{R}^N)$ be the product space of $\mathbb{R}^N$ and the space of all $(N-1)$-dimensional unoriented subspaces in $\mathbb{R}^N$. We say that V is a $(N-1)$-*varifold* in $\mathbb{R}^N$ if V is a Radon measure on $G_{N-1}(\mathbb{R}^N)$, or equivalently a non-negative linear functional on $G_{N-1}(\mathbb{R}^N)$. We call a varifold V an *integer varifold* if there exists a countably $(N-1)$-rectifiable set M and a non-negative $\mathscr{H}^{N-1}$-locally integrable function θ with $\theta(\cdot) \in \mathbb{N}$ such that

$$V(\phi) = \int_M \theta(x)\phi(x, T_xM)\, d\mathscr{H}^{N-1}(x) \quad \text{for } \phi \in C_c(G_{N-1}(\mathbb{R}^N)).$$

The function θ is called the multiplicity function.

For a varifold V, we define the associated weight measure $\|V\|$ as

$$\|V\|(\phi) := \int_{G_{N-1}(\mathbb{R}^N)} \phi(x)\, dV(x, S) \quad \text{for } \phi \in C_c(\mathbb{R}^N),$$

and the first variation δV is defined as a linear functional on $C_c(\mathbb{R}^N, \mathbb{R}^N)$:

$$\delta V(f) := \int_{G_{N-1}(\mathbb{R}^N)} \operatorname{div}_S\, f(x)\, dV(x, S) \quad \text{for} f \in C_c^1(\mathbb{R}^N, \mathbb{R}^N)$$

where $\operatorname{div}_S\, f(x)$ denotes the tangential divergence i.e.
$\operatorname{div}_S\, f(x) := \operatorname{trace}(p_S \nabla f(x))$, with p_S is the orthogonal projection onto S.

We now assume δV is locally bounded ; then δV has a uniquely linear extension on $C_c(\mathbb{R}^N, \mathbb{R}^N)$. The Riesz representation theorem establishes the existence of a Radon measure, denoted by $\|\delta V\|$ and a $\|\delta V\|$-measurable function $\boldsymbol{n} : \mathbb{R}^N \to \mathbb{R}^N$ with $|\boldsymbol{n}(x)| = 1$ $\|\delta V\|$-a.e. x, and for any $g \in C_c(\mathbb{R}^N; \mathbb{R}^N)$,

$$\delta V(g) = -\int_{\mathbb{R}^N} g \cdot \boldsymbol{n}\, d\|\delta V\|.$$

Furthermore, assuming that $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$, we define the *generalised mean curvature vector* $\boldsymbol{h}$ of the $(N-1)$-varifold V as follows:

$$\delta V(f) = \int_{\mathbb{R}^N} f \cdot \boldsymbol{h}\, d\|V\| \quad \text{for} f \in C_c^1(\mathbb{R}^N, \mathbb{R}^N).$$

The Radon-Nikodym derivative $D_{\|V\|}\|\delta V\|(x)$ of $\|\delta V\|$ with respect to $\|V\|$, exists for $\|V\|$- a.e. x, and we can express $\boldsymbol{h}(x) = D_{\|V\|}\|\delta V\|(x)\boldsymbol{n}(x)$. It is known that $\boldsymbol{h}(x)$ is perpendicular to the tangent space T_xV for an integer $(N-1)$-varifold (see [5, Chapter 5]). In this sense, we can regard $\boldsymbol{n}$ as the generalised outer unit normal vector.

(iii) Caccioppoli sets

We recall the distributional idea of the mean curvature and the normal vector in the framework of Caccioppoli sets from [11]. We say that a Lebesgue measurable set E in $\mathbb{R}^N$ is a Caccioppoli set (or set of locally finite perimeter) in $\mathbb{R}^N$ if the characteristic function χ_E belongs to $\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^N)$, that is, the distributional derivative $D\chi_E$ is a $\mathbb{R}^N$-valued Radon measure on $\mathbb{R}^N$. Then, we can define the total variation measure $|D\chi_E|$, which satisfies

$$|D\chi_E|(A) = \sup\Big\{ \int_E \mathrm{div}_E f \, d\mathscr{L}^N : f \in C_c^1(A;\mathbb{R}^N), \|f\|_\infty \leq 1 \Big\}$$

for every open set $A \subset \mathbb{R}^N$. The reduced boundary $\partial^* E$ of a Caccioppoli set E in $\mathbb{R}^N$ is the set of those $x \in \operatorname{spt} D\chi_E$ such that the limit

$$\nu_E(x) := \lim_{r \to 0+} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))} \quad \text{exists and belongs to} \quad \mathbb{S}^{N-1},$$

and we call ν_E the measure-theoretic outer unit normal to E. By De Giorgi's structure theorem ([11, Theorem15.9]), we have $D\chi_E = \nu_E \mathscr{H}^{N-1} \lfloor \partial^* E$.

We say that E has the distributional mean curvature if there exists $\kappa \in L^1_{\mathrm{loc}}(\partial^* E, \mathscr{H}^{N-1})$ such that

$$\int_{\partial^* E} \mathrm{div}_E f \, d\mathscr{H}^{N-1} = \int_{\partial^* E} (f \cdot \nu_E)\kappa \, d\mathscr{H}^{N-1} \quad (f \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N))$$

where $\mathrm{div}_E f(x) := \operatorname{div} f(x) - \nu_E(x) \cdot \nabla f(x) \nu_E(x) \quad (x \in \partial^* E)$ is the tangential divergence of f on $\partial^* E$. If E is a Caccioppoli set with the distributional mean curvature κ, then we can define the $(N-1)$-varifold with multiplicity 1, denoted by $V := \mathrm{var}(E, 1)$ as follows.

$$\mathrm{var}(E,1)(\phi) = \int_E \phi(x, \pi_x) \, d\mathscr{H}^{N-1}(x) \quad \text{for } \phi \in C_c(G_{N-1}(\mathbb{R}^N))$$

where $\pi_x := \nu_E(x)^\perp$. In this case, since $\delta V(f) = \int_{\mathbb{R}^N} \mathrm{div}_{\pi_x} f \, d\mathscr{H}^{N-1} \lfloor \partial^* E = \int_{\partial^* E} \mathrm{div}_E f \, d\mathscr{H}^{N-1}$, $\|V\| = \mathscr{H}^{N-1} \lfloor \partial^* E$, by the definition of $\boldsymbol{h}$, we can observe that $\boldsymbol{h} = \kappa \nu_E$.

1 Weak formulation of acceleration

Varifold formulation

Firstly, we recall that the normal velocity is characterised by the variation of the weighted surface area, that is,

$$\frac{d}{dt} \int_{\Gamma_t} \phi \, d\mathscr{H}^{N-1} = \int_{\Gamma_t} \Big[v(\nabla \phi \cdot \boldsymbol{n}) + \phi_t - \phi v \kappa \Big] \, d\mathscr{H}^{N-1}, \tag{4}$$

for $\phi \in C_c^1(\mathbb{R}^N \times [0,\infty))$ under the assumption that the moving hypersurface $\mathcal{M}$ is C^1 and its unit normal vector field is C^1 on $\mathcal{M}$. In particular, each Γ_t is a C^2-hypersurface and depends C^1-smoothly on time. Additionally, we assume that Γ_t is compact for all $t \geq 0$.

The idea of characterising the normal velocity via the integral formula, is due to Brakke [5]. More precisely, it is known that the inequality obtained by replacing the equality (4) with "$\leq$" is sufficient to characterise the normal velocity (see [19] for more details). In a related context, Mugnai and Röger [12] have also proposed a notion of weak velocity for the Radon measure, specifically for L^2-flows, which provides a complementary perspective to our approach. In the case of the acceleration, we consider the time derivative of

$$\int_{\Gamma_t} v\,\phi\, d\mathscr{H}^{N-1} \quad \text{for } \phi \in C_c^1(\mathbb{R}^N \times [0,\infty)).$$

This quantity can be regarded as the weighted normal momentum of Γ_t. Now we show the following variational formula of the acceleration $D_t v$, which is an analogue of [15, Proposition 5.2.1].

Proposition 1 (Variational identity for acceleration)**.** *Let $\mathcal{M} = \bigcup_{t\geq 0}\Big(\Gamma_t \times \{t\}\Big)$ be a moving hypersurface in $\mathbb{R}^N$. Assume that Γ_t is compact for all $t \geq 0$, $v \in C^1(\mathcal{M})$. Then, for $\phi \in C_c^1(\mathbb{R}^N \times [0,\infty))$,*

$$\int_{\Gamma_t} v\phi\, d\mathscr{H}^{N-1}\Big|_{t=t_1}^{t_2} = \int_{t_1}^{t_2}\int_{\Gamma_t}\Big[(D_t v - v^2\kappa)\phi + v(v\nabla\phi\cdot\boldsymbol{n} + \phi_t)\Big]\, d\mathscr{H}^{N-1}\, dt, \tag{5}$$

for $t_1, t_2 \in [0,\infty)$ with $t_1 < t_2$.

Proof Since $\phi \in C_c^1(\mathbb{R}^N \times [0,\infty))$, we extend it to a C^1-function on an open neighborhood $\mathcal{Q}$ of $\mathbb{R}^N \times [0,\infty)$ in $\mathbb{R}^{N+1}$. By abuse of notation, we still denote this extension by ϕ, and also its restriction to $\mathcal{M}$. Here, we remark that v is defined only on $\mathcal{M}$ (and not on $\mathcal{Q}$). Thus, we can apply the transport identity (2) for $v\phi \in C^1(\mathcal{M})$ to get

$$\frac{d}{dt}\int_{\Gamma_t} v\phi\, d\mathscr{H}^{N-1} = \int_{\Gamma_t}\{D_t(v\phi) - (v\phi)v\kappa\}\, d\mathscr{H}^{N-1}.$$

The product formula for the normal time derivative yields $D_t(v\phi) = \phi D_t v + vD_t\phi$ and the calculus for $D_t\phi$ gives

$$\frac{d}{dt}\int_{\Gamma_t} v\phi\, d\mathscr{H}^{N-1} = \int_{\Gamma_t}\Big[\phi D_t v + v(v\nabla\phi\cdot\boldsymbol{n} + \phi_t) - v^2\phi\kappa\Big]\, d\mathscr{H}^{N-1}.$$

Integrating this equation over $[t_1, t_2]$ concludes the proof. $\boxed{QED}$

The following two remarks were also mentioned in the author's doctoral dissertation [15], but is reproduced here for the sake of self-containment.

Remark 1. For equality (5) to be meaningful, it is sufficient that the normal velocity and the mean curvature are locally 3-th power integrable with respect to $\mathscr{H}^{N-1}\lfloor_{\Gamma_t}\times dt$. Here, we say that the function $f = f(x,t)$ $(x,t) \in \mathcal{M}$, is locally p-th power integrable with respect to $\mathscr{H}^{N-1}\lfloor_{\Gamma_t}\times dt$ if for any compact set $K \subset \mathbb{R}^{N+1}$, $\int_{K\cap\mathcal{M}} |f|^p \, d\mathscr{H}^{N-1}\, dt < \infty$. If so, by applying Hölder inequality with $p = 3/2$. $q = 3$, we get

$$
\begin{aligned}
&\int_{t_1}^{t_2}\int_{\Gamma_t} \phi v^2\kappa \, d\mathscr{H}^{N-1}\, dt \le L \int_{\text{spt }\phi\cap\mathcal{M}} |v^2\kappa| \, d\mathscr{H}^{N-1}\, dt \\
&\le L\Big(\int_{\text{spt }\phi\cap\mathcal{M}} |v^2|^p \, d\mathscr{H}^{N-1}\, dt\Big)^{\frac{1}{p}}\Big(\int_{\text{spt }\phi\cap\mathcal{M}} |\kappa|^q \, d\mathscr{H}^{N-1}\, dt\Big)^{\frac{1}{q}} \\
&= L\Big(\int_{\text{spt }\phi\cap\mathcal{M}} |v|^3 \, d\mathscr{H}^{N-1}\, dt\Big)^{\frac{2}{3}}\Big(\int_{\text{spt }\phi\cap\mathcal{M}} |\kappa|^3 \, d\mathscr{H}^{N-1}\, dt\Big)^{\frac{1}{3}},
\end{aligned}
$$

where $L := \sup_{\mathbb{R}^N\times[0,\infty)} |\phi| < \infty$. Also, the other quantities

$$
\int_{t_1}^{t_2}\int_{\Gamma_t} v\, D_t v\, \phi \, d\mathscr{H}^{N-1}\, dt, \qquad \int_{t_1}^{t_2}\int_{\Gamma_t} v^2(v\nabla\phi\cdot\boldsymbol{n} + \phi_t)\, d\mathscr{H}^{N-1}\, dt
$$

are finite under these assumptions and $D_t v$ is locally integrable.

Remark 2. By using formula (5), we can characterize the acceleration $D_t v$ in the following sense. Suppose that a function $\tilde{a} : \mathcal{M} \to \mathbb{R}$ satisfies the following identity,

$$
\int_{\Gamma_t} v\phi \, d\mathscr{H}^{N-1}\Big|_{t=t_1}^{t_2} = \int_{t_1}^{t_2}\int_{\Gamma_t}\Big[(\tilde{a} - v^2\kappa)\phi + v(v\nabla\phi\cdot\boldsymbol{n} + \phi_t) - \phi v^2\kappa\Big]\, d\mathscr{H}^{N-1}\, dt,
$$

by combining with (5), we have

$$
\int_{t_1}^{t_2}\int_{\Gamma_t}(\tilde{a} - D_t v)\phi \, d\mathscr{H}^{N-1}\, dt = 0
$$

for all test functions $\phi \in C_c^1(\mathbb{R}\times[0,\infty))$. Then, we can get

$$
\tilde{a} = D_t v \;\; \mathscr{H}^{N-1}\lfloor_{\Gamma_t}\times dt\text{-a.e. on} \quad \mathcal{M}.
$$

Motivated by Proposition 1, let us define the generalised notion of the acceleration $D_t v$ for a family of varifolds. We recall the notion of weak velocity vector for varifolds. See [19] for precise definitions. Let $\{V_t\}_{t\in[0,T)}$, $T \le \infty$ be a family of integral $(N-1)$-varifolds in $\mathbb{R}^N$ with the generalized mean curvature vector $\boldsymbol{h}(\cdot,t)$. The vector valued function $\boldsymbol{v} \in L^2_{\text{loc}}(\|V_t\|\times dt)$ is the weak normal velocity vector of $\{V_t\}_{t\in[0,T)}$, if $\boldsymbol{v}$ satisfies for V_t-a.e. $(x,S)\in G_{N-1}(\mathbb{R}^N)$, $p_S(\boldsymbol{v}(x,t))=0$ where p_S is the orthogonal projection to S, and for all $0\le t_1<t_2<T$, for any $\phi\in C^1_c(\mathbb{R}^N\times[0,T)\,;\,\mathbb{R}_{\ge 0})$ (here, following Brakke's definition, we require that ϕ be nonnegative.),

$$\int_{\mathbb{R}^N}\phi\, d\|V_t\|\bigg|_{t=t_1}^{t_2}\le\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\Big[(\nabla\phi-\phi\boldsymbol{h})\cdot\boldsymbol{v}+\phi_t\Big]\, d\|V_t\|\, dt$$

It is known that $T_xV_t\perp\boldsymbol{v}(x,t)$. For the family of integer varifolds $\{V_t\}_{t\in[0,T)}$ with the weak normal velocity $\boldsymbol{v}$, we define $\tilde\kappa,\,\tilde v:\mathbb{R}^N\times[0,T)\to\mathbb{R}$ as follows.

$$\tilde\kappa(x,t):=D_{\|V_t\|}\|\delta V_t\|(x),\quad \tilde v(x,t):=\boldsymbol{v}(x,t)\cdot\boldsymbol{n}(x,t)$$

Now, we attain the definition of the acceleration of the varifold. In the following definition, as we mentioned before, to ensure the finiteness of integrals appearing on the right-hand side, we need locally 3-th power integrability of v,κ.

Definition 1 (Weak acceleration of a varifold)**.** Let $\{V_t\}_{t\in[0,T)}$, $T\le\infty$ be a family of $(N-1)$-varifolds in $\mathbb{R}^N$ with $\tilde\kappa,\,\tilde v\in L^3_{\mathrm{loc}}(\|V_t\|\times dt)$. Then, $a_{\mathrm{var}}\in L^1_{\mathrm{loc}}(\|V_t\|\times dt)$ is called the weak acceleration if a satisfies the following equality:

$$\int_{\mathbb{R}^N}\tilde v\phi\, d\|V_t\|\bigg|_{t=t_1}^{t_2}=\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\Big[(a_{\mathrm{var}}-\tilde v^2\tilde\kappa)\phi+\tilde v(\nabla\phi\cdot\boldsymbol{v}+\phi_t)\Big]\, d\|V_t\|\, dt.\qquad(6)$$

for all $\phi\in C^\infty_c(\mathbb{R}^N\times[0,T))$, for $t_1,t_2\in[0,T)$ with $t_1<t_2$.

By the same argument of Remark 2, the weak acceleration is determined uniquely. It coincides with D_tv in the case that $V_t=|\Gamma_t|$ with C^1-class $(N-1)$-dimensional hypersurfaces Γ_t with smooth $\boldsymbol{n}$.

Caccioppoli sets formulation

The second formulation is based on the relation between ϕ_{tt} and D_tv. We also recall the characterisation of the normal velocity v for Caccioppoli sets following [17](also known as BV flow, see [18]). Let Ω_t be bounded open sets with smooth boundary Γ_t for all $t\ge 0$. We apply the transport identity (3) for volume with $f=\phi\in C^2_c(\mathbb{R}^N\times[0,T))$. This is justified since, for any $\phi\in C^2_c(\mathbb{R}^N\times[0,T))$, the extension argument uesd in the proof of Proposition 1 implies that $\phi\in C^1(\mathcal{M})$ and $\phi\in C^1(\overline{Q})$ $(Q:=\bigcup_{t\ge0}(\Omega_t\times\{t\}))$. Thus, we can apply the transport identity (3) for ϕ, and integrate over (t_1,t_2) for $t_1,t_2\in[0,T)$ to obtain

$$\int_{\Omega_t}\phi\, d\mathscr{L}^N\bigg|_{t=t_1}^{t_2}=\int_{t_1}^{t_2}\int_{\Omega_t}\phi_t\, d\mathscr{L}^N\, dt-\int_{t_1}^{t_2}\int_{\Gamma_t}\phi v d\mathscr{H}^{N-1}\, dt.\qquad(7)$$

Motivated by (7), the familiy of Caccioppoli sets $\{E_t\}$ moves with the weak normal velocity v if there exists $v \in L^2_{\mathrm{loc}}(\mathscr{H}^{N-1}\lfloor\partial^* E_t \times dt)$ such that for any $\phi \in C^1_c(\mathbb{R}^N \times [0,T))$, it holds

$$\int_{E_t} \phi \, d\mathscr{L}^N \bigg|_{t=t_1}^{t_2} = \int_{t_1}^{t_2} \int_{E_t} \phi_t \, d\mathscr{L}^N \, dt - \int_{t_1}^{t_2} \int_{\partial^* E_t} \phi v d\mathscr{H}^{N-1} \, dt. \tag{8}$$

We now consider the time derivative of $\int_{\Omega_t} \phi_t \, d\mathscr{L}^N$ for $\phi \in C^2_c(\mathbb{R}^N \times [0,T))$. As in the same argument above, applying the transport identity (3) for ϕ_t and integrating over (t_1, t_2) on both sides, we have

$$\int_{\Omega_t} \phi_t \, d\mathscr{L}^N \bigg|_{t=t_1}^{t_2} = \int_{t_1}^{t_2} \int_{\Omega_t} \phi_{tt} \, d\mathscr{L}^N \, dt - \int_{t_1}^{t_2} \int_{\Gamma_t (=\partial\Omega_t)} \phi_t v d\mathscr{H}^{N-1} \, dt. \tag{9}$$

On the other hand, applying Proposition 1 to have

$$\begin{aligned} &\int_{t_1}^{t_2} \int_{\Gamma_t} \phi_t v \, d\mathscr{H}^{N-1} \, dt \\ &= \int_{\Gamma_t} \phi v \, d\mathscr{H}^{N-1} \bigg|_{t=t_1}^{t_2} - \int_{t_1}^{t_2} \int_{\Gamma_t} \Big[(D_t v - v^2 \kappa)\phi + v^2 \nabla\phi \cdot \boldsymbol{n} \Big] \, d\mathscr{H}^{N-1} \, dt \end{aligned} \tag{10}$$

By combining identities (9),(10), we attain the following identity:

$$\begin{aligned} &\int_{\Omega_t} \phi_t \, d\mathscr{L}^N + \int_{\Gamma_t} \phi v \, d\mathscr{H}^{N-1} \bigg|_{t=t_1}^{t_2} \\ &= \int_{t_1}^{t_2} \int_{\Omega_t} \phi_{tt} \, d\mathscr{L}^N \, dt + \int_{t_1}^{t_2} \int_{\Gamma_t} \Big[(D_t v - v^2 \kappa)\phi + v^2 \nabla\phi \cdot \boldsymbol{n} \Big] \, d\mathscr{H}^{N-1} \, dt \end{aligned} \tag{11}$$

Motivated by (11), let us define the generalised notion of the acceleration $D_t v$ for the family of Caccioppoli sets.

Definition 2 (Weak acceleration for Caccioppoli sets)**.** Let $\{E_t\}_{t\in[0,T)}$, $T \leq \infty$ be a family of Caccioppoli sets in $\mathbb{R}^N$ with the weak normal velocity $v \in L^3_{\mathrm{loc}}(\mathscr{H}^{N-1}\lfloor\partial^* E_t \times dt)$, and $\kappa \in L^3_{\mathrm{loc}}(\mathscr{H}^{N-1}\lfloor\partial^* E_t \times dt)$. Then, $a_{\mathrm{cac}} \in L^1_{\mathrm{loc}}(\mathscr{H}^{N-1}\lfloor\partial^* E_t \times dt)$ is called the weak acceleration if a_{cac} satisfies the following equality:

$$\begin{aligned} &\int_{E_t} \phi_t \, d\mathscr{L}^N + \int_{\partial^* E_t} v\phi \, d\mathscr{H}^{N-1} \bigg|_{t=t_1}^{t_2} \\ &= \int_{t_1}^{t_2} \int_{E_t} \phi_{tt} \, d\mathscr{L}^N \, dt + \int_{t_1}^{t_2} \int_{\partial^* E_t} \Big[(a_{\mathrm{cac}} - v^2 \kappa)\phi + v^2 \nabla\phi \cdot \nu_{E_t} \Big] \, d\mathscr{H}^{N-1} \, dt \end{aligned} \tag{12}$$

for all $\phi \in C^\infty_c(\mathbb{R}^N \times [0,T))$, for $t_1, t_2 \in [0,T)$ with $t_1 < t_2$

By the same argument of the previous section, that is, the case of the varifold, the weak acceleration for Caccioppoli sets is determined uniquely.

We now illustrate the above two notions of weak acceleration by a simple example.

Example 1 (Evolving spheres)**.** Let $B_{R(t)} \subset \mathbb{R}^N$ be the ball of radius $R(t)$, and let $\Gamma_t := \partial B_{R(t)}$ for $t \geq 0$. Then, for $x \in \Gamma_t$, we have

$$\boldsymbol{n}(x,t) = \frac{x}{R(t)} \left(= \nu_{B_{R(t)}}(x) \right), \quad \kappa(x,t) = -\frac{N-1}{R(t)},$$

and

$$v(x,t) = R'(t), \quad D_t v(x,t) = R''(t).$$

Then, the equation (1) with initial conditions becomes

$$\begin{cases} R''(t) &= -\dfrac{N-1}{R(t)}, \\ R(0) &= R_0 > 0, \\ R'(0) &= v_0 \in \mathbb{R}. \end{cases}$$

A direct computation shows that these quantities satisfy (6) and (12). Thus, the two notions of weak acceleration coincide and are given by

$$a_{\mathrm{var}}(x,t) = a_{\mathrm{cac}}(x,t) = R''(t).$$

This observation is generalized in the next section.

2 Relationship between two weak accelerations

In this final section, we investigate the relationship between the two weak accelerations. We have the following theorem.

Theorem 1. *Let $\{E_t\}_{t\in[0,T)}$ be a family of Caccioppoli sets in $\mathbb{R}^N$ with weak mean curvature κ , normal velocity v, acceleration a_{cac}. Assume that the weak velocity vector $\boldsymbol{v}$ for $V_t := \mathrm{var}(E_t, 1)$ satisfies $\boldsymbol{v}(x,t) = v(x,t)\nu_{E_t}(x)$. Then, $a_{\mathrm{cac}}(x,t) = a_{\mathrm{var}}(x,t)$ for $(\mathscr{H}^{N-1} \times dt)$-almost every (x,t).*

Proof We will show for any $\phi \in C_c^\infty(\mathbb{R}^N \times [0,T))$, for $t_1, t_2 \in [0,T)$ with $t_1 < t_2$.

$$\int_{\mathbb{R}^N} \tilde{v}\phi \, d\|V_t\| \Big|_{t=t_1}^{t_2} = \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \left[(a_{\mathrm{cac}} - \tilde{v}^2\tilde{\kappa})\phi + \tilde{v}(\nabla\phi \cdot \boldsymbol{v} + \phi_t) \right] d\|V_t\| \, dt \quad (13)$$

where $\tilde{v}, \tilde{\kappa}$ denote the weak velocity, mean curvature for varifold V_t respectively, defined in the preceding sections. Firstly, as mentioned in the Introduction, since

$\boldsymbol{h}(\cdot,t) = \tilde{\kappa}(\cdot,t)\boldsymbol{n}(\cdot,t) = \kappa(\cdot,t)\nu_{E_t}(\cdot)$ and $\boldsymbol{v}(\cdot,t) = \tilde{v}(\cdot,t)\boldsymbol{n}(\cdot,t) = v(\cdot,t)\nu_{E_t}(\cdot)$ we have $\tilde{\kappa} = \kappa$, $\boldsymbol{n} = \nu_{E_t}$, $\tilde{v} = v$. Thus, the identity (13) is rewritten as follows:

$$\int_{\mathbb{R}^N} v\phi \, d\|V_t\| \Big|_{t=t_1}^{t_2} = \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \Big[(a_{\mathrm{cac}} - v^2\kappa)\phi + v(\nabla\phi \cdot \boldsymbol{v} + \phi_t) \Big] \, d\|V_t\| \, dt. \quad (14)$$

By the definition (12) of a_{cac}, we have

$$\begin{aligned} \int_{\partial^* E_t} v\phi \, d\mathscr{H}^{N-1} \Big|_{t=t_1}^{t_2} &= \int_{t_1}^{t_2} \int_{E_t} \phi_{tt} \, d\mathscr{L}^N \, dt - \int_{E_t} \phi_t \, d\mathscr{L}^N \Big|_{t=t_1}^{t_2} \\ &\quad + \int_{t_1}^{t_2} \int_{\partial^* E_t} \Big[(a_{\mathrm{cac}} - v^2\kappa)\phi + v^2 \nabla\phi \cdot \nu_{E_t} \Big] \, d\mathscr{H}^{N-1} \, dt. \end{aligned} \quad (15)$$

On the other hand, applying (8) to ϕ_t to get

$$\int_{E_t} \phi_t \, d\mathscr{L}^N \Big|_{t=t_1}^{t_2} = \int_{t_1}^{t_2} \int_{E_t} \phi_{tt} \, d\mathscr{L}^N \, dt - \int_{t_1}^{t_2} \int_{\partial^* E_t} \phi_t v d\mathscr{H}^{N-1} \, dt. \quad (16)$$

Substituting (16) into (15), we have

$$\begin{aligned} &\int_{\partial^* E_t} v\phi \, d\mathscr{H}^{N-1} \Big|_{t=t_1}^{t_2} \\ &= \int_{t_1}^{t_2} \int_{\partial^* E_t} \phi_t v d\mathscr{H}^{N-1} \, dt + \int_{t_1}^{t_2} \int_{\partial^* E_t} \Big[(a_{\mathrm{cac}} - v^2\kappa)\phi + v^2 \nabla\phi \cdot \nu_{E_t} \Big] \, d\mathscr{H}^{N-1} \, dt. \\ &= \int_{t_1}^{t_2} \int_{\partial^* E_t} \Big[(a_{\mathrm{cac}} - v^2\kappa)\phi + v(v\nabla\phi \cdot \nu_{E_t} + \phi_t) \Big] \, d\mathscr{H}^{N-1} \, dt. \end{aligned} \quad (17)$$

Since $\|V_t\| = \mathscr{H}^{N-1} \lfloor E_t$ and $v\nu_{E_t} = \boldsymbol{v}$, (17) implies (14), then we get (13). By combining the definition of a_{var} with (13), we obtain

$$\int_{t_1}^{t_2} \int_{\partial^* E_t} a_{\mathrm{cac}}\phi \, d\mathscr{H}^{N-1} \, dt = \int_{t_1}^{t_2} \int_{\partial^* E_t} a_{\mathrm{var}}\phi \, d\mathscr{H}^{N-1} \, dt \quad (18)$$

Since ϕ and t_1, t_2 are arbitrary, we get the conclusion. $\boxed{QED}$

Conclusion

This note provided two weak notions of acceleration for varifold and Caccioppoli set, respectively. These formulations provide a new foundation for studying weak solutions to mean curvature accelerated flow. Future tasks remain the construction of the weak solution for the mean curvature accelerated flow, and applying the framework to numerical schemes.

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