Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **46** (2026) no. 1, 19–40. doi:10.1285/i15900932v46n1p19-40

Generalizing Eulerian Numbers via Semipermutations: Topological and Combinatorial Aspects

Giovanni Gaiffi
Università di Pisa
Dipartimento di Matematica
Largo Bruno Pontecorvo 5, 56127 Pisa
Italy
giovanni.gaiffi@unipi.it

Giovanni Interdonato
Scuola Normale Superiore
Piazza dei Cavalieri 7, 56126 Pisa
Italy
giovanni.interdonato@sns.it

Received: 10.02.26; accepted: 25.05.26.

Abstract. In a paper by Lin an interesting family of semipermutations comes out to index the elements of a cohomology basis of a Hessenberg type variety. The corresponding Betti numbers are a generalization of Eulerian numbers. We show three different subsets of the symmetric group that are in bijection with the set of these semipermutations. These bijections preserve the statistics lec and des: one of these is first obtained by an algebraic-topological argument, and all of them are explicitly described in combinatorial terms.

Keywords: Eulerian numbers, semipermutations, Hessenberg varieties, wonderful models

MSC 2020 classification: primary 05E14, secondary 05A05

Introduction

In [7] the author points out the following family of semipermutations, more precisely the set $\mathcal{S}_{n,k+1}$ of permutations of $(k+1)$ different numbers chosen from $[n] := \{ 1, \cdots , n \}$, equipped with a descent statistic.

Definition 1. The set $\mathcal{S}_{n,k+1}$ is made of all pairs (α_0, ρ), where $\rho = \rho_1 \cdots \rho_{k+1}$ is a word made by $k+1$ different numbers of $[n]$, and $\alpha_0 = [n] \setminus \{ \rho_1, \ldots , \rho_{k+1} \}$.

Given a pair (α_0, ρ) as above, we define its *descent set* as

$$\mathrm{Des}((\alpha_0, \rho)) = \{ (w_i > w_{i+1}) \mid i \in [k] \} \sqcup \{ (a > \rho_1) \mid a \in \alpha_0 \}$$

and we will denote by $\mathrm{des}((\alpha_0, \rho)) = |\,\mathrm{Des}((\alpha_0, \rho))|$ the cardinality of this set.

Example 1. Let $(\alpha_0, \tau) = (\{4, 6, 7\}, 52318) \in \mathcal{S}_{8,5}$. We have that

$$\mathrm{Des}((\{4, 6, 7\}, 52318)) = \{(5 > 2), (3 > 1)\} \sqcup \{(6 > 5), (7 > 5)\}.$$

Therefore $\mathrm{des}((\{4, 6, 7\}, 52318)) = 2 + 2 = 4$.

The number of permutations of $k+1$ different numbers in $[n]$ with i descents appears naturally in [7] as the $2i$-th Betti number of a Hessenberg type variety. The author notices that these Betti numbers are a natural generalization of the Eulerian numbers (when $k = n-2$ we are dealing with the permutations of $n-1$ different numbers in $[n]$ with i descents, which is equivalent to dealing with the permutations of n numbers with i descents, whose cardinality is the Eulerian number $A(n, i+1)$) and observes that it would be interesting to study further properties of these numbers (see also this related question on MathOverflow [1] and the OEIS sequence [8]).

In the present paper we explore this topic, providing a further topological interpretation of these numbers. This gives the inspiration to then find three explicit bijections of the set of permutations of $k + 1$ different numbers in $[n]$ with i descents with some sets of permutations considered with the statistics lec or des. One of the involved bijections turns out to be an explicit presentation of a bijection introduced by Foata and Han in [3].

The structure of this paper is the following. After recalling in Section 1 the statistics des and lec, in Section 2 we consider the Hessenberg type variety whose cohomology in degree $2i$, according to Lin's construction, admits a basis indexed by the permutations of $k+1$ different numbers in $[n]$ with i descents. We notice that this variety is isomorphic to a De Concini–Procesi wonderful model (these models were introduced in the seminal paper [2]): a different construction of a cohomology basis coming from the theory of wonderful models then provides us with an algebraic-topological proof of a bijection between the set of the permutations of $k + 1$ different numbers in $[n]$ with i descents and the set of the elements of $\mathcal{S}_n$ whose first hook from the right in the hook factorization has cardinality at least $n - k$ and whose lec is equal to i.

Motivated by this, we show an explicit combinatorial presentation of the above mentioned bijection. We do this in two steps. First in Section 3 we deal with the case $k = n - 2$, i.e. we provide a bijection of S_n which preserves the statistics lec and des. This is based on the idea of *reverse wave*: let $w = w_1 \cdots w_\ell$ be a non-empty word made of distinct numbers. Let c be the position in w of the maximum element of w, then w is a *reverse wave* if $c \neq \ell$ and

$$w_1 < w_2 < \cdots < w_{c-1} < w_\ell < w_{\ell-1} < \cdots < w_c. \tag{1}$$

or if $c = \ell$ and the elements are in increasing order.

Then in Section 4 we extend this bijection to the general case (i.e. for any value of k).

Finally, in Section 5 we present a different bijection ψ of the symmetric group $\mathcal{S}_n$ that sends the des statistic to the lec statistic. This bijection ψ is explicitly presented and appears to be equal, unless we compose with easy bijections, to the map F defined by Foata and Han in [3, Section 6]). It is based on the idea of *wave*: let $w = w_1 \cdots w_\ell$ be a non-empty word made of distinct numbers. Let c be the position in w of the maximum element of w, we say that w is a *wave* if $c \neq \ell$ and

$$w_\ell < w_{\ell-1} < \cdots < w_{c+1} < w_1 < w_2 < \cdots < w_c. \tag{2}$$

or if $c = \ell$ and the elements are in increasing order.

Also this bijection can be extended to the general case so in the end of the section we obtain Theorem 6 which shows three different sets of S_n whose cardinality is equal to the cardinality of the set of the permutations of $k+1$ different numbers in $[n]$ with i descents. This extends our first algebraic-topological result, and therefore gives an explicit insight on the semipermutations pointed out by Lin.

1 Some preliminary notation

If $w = w_1 \dots w_\ell$ is a word, we define its descent set as $\mathrm{Des}(w) = \{\, (w_i > w_{i+1}) \mid i \in [\ell - 1] \,\}$ and we will denote by $\mathrm{des}(w) = |\,\mathrm{Des}(w)|$ the cardinality of this set. We define its inversion set as $\mathrm{Inv}(w) = \{\, (i, j) \mid 1 \le i < j \le \ell,\, w_i > w_j \,\}$ and we will denote by $\mathrm{inv}(w) = |\,\mathrm{Inv}(w)|$ the cardinality of this set. These definitions naturally extend to permutations, considering the word obtained by writing them in one-line notation.

Example 2. Let $\sigma = 45762318 \in \mathcal{S}_8$. We will have that $\mathrm{Des}(\sigma) = \{\, (7 > 6), (6 > 2), (3 > 1) \,\}$ and hence that $\mathrm{des}(\sigma) = 3$.

Given a word $w = w_1 \cdots w_\ell$ we recall the definition of Hook factorization, first introduced by Gessel in [6]. There exists and is unique a decomposition of $w = \gamma * \alpha_1 * \cdots * \alpha_k$, where the operator $*$ is the concatenation of words, such that the elements in γ are in increasing order, and each α_i is a *hook*, that is $\alpha_i = a_1^{(i)} \cdots a_{s_i}^{(i)}$, with $s_i \ge 2$, $a_1^{(i)} > a_2^{(i)}$ and $a_2^{(i)} < \cdots < a_{s_i}^{(i)}$, for each $i = 1, \dots, k$. We then define the statistic $\mathrm{lec}(w) = \displaystyle\sum_{i=1}^{k} \mathrm{inv}(\alpha_i)$ as the sum of the number of inversions of each hook. We will also denote a word with elements in increasing order, such as γ, as a *trivial hook*.

Remark 1. We can obtain this factorization recursively, taking as α_k the suffix of w that starts at the position of the rightmost descent of w, and then iterating this reasoning on the remaining prefix obtained by removing α_k. The process ends when no further descents are present, i.e. when we have a word whose elements are in increasing order: this will be γ.

Example 3. Let $\tau = 18326457 \in \mathcal{S}_8$. Its Hook factorization is:

$$\tau = \underline{18} * 32 * 6457, \tag{3}$$

where $\gamma = 18$ is the underlined factor. We therefore have that $\mathrm{lec}(\tau) = \mathrm{inv}(32) + \mathrm{inv}(6457) = 1 + 2 = 3$.

2 A bijection pointed out by algebraic topology

Let us denote by $\mathcal{X}$ the permutohedral variety of dimension $n-1$, i.e. the toric variety associated to the root system of type A_{n-1}. As it is well known (see for instance [9],[7, Section 2]), it can also be obtained as an iterated blowup of $\mathbb{P}^{n-1}$:

$$\mathcal{X} = \mathcal{X}_{n-2} \xrightarrow{\pi_{n-2}} \mathcal{X}_{n-3} \xrightarrow{\pi_{n-3}} \cdots \xrightarrow{\pi_2} \mathcal{X}_1 \xrightarrow{\pi_1} \mathcal{X}_0 = \mathbb{P}^{n-1}$$

where each $\mathcal{X}_{k+1}$ is the blowup of $\mathcal{X}_k$ along the strict transform of the k-dimensional coordinate spaces ($0 \leq k \leq n-3$).

In [7, Section 3] the author describes an explicit isomorphism between the permutohedral variety of dimension $n-1$, and the regular semisimple Hessenberg variety $\mathcal{H}ess(\mathbf{S}, h_+)$ associated to the Hessenberg function h_+ defined by $h_+(i) = i+1$ for $1 \leq i \leq n-1$ and $h_+(n) = n$. In this process, he further obtains isomorphisms from $\mathcal{X}_k$, the $n-1$ dimensional variety that is the k-th step in the iterated blowups, to a "Hessenberg type" subvariety of a partial flag variety which is denoted by $\mathcal{H}ess^{(k+1)}(\mathbf{S}, h_+)$. Then he exhibits a basis for the integer cohomology $H^*(\mathcal{X}_k)$ whose elements are in one-to-one correspondence with the permutations of $(k+1)$ different numbers chosen from $[n] := \{1, \cdots, n\}$. This allows for a combinatorial computation of the Betti numbers of $\mathcal{X}_k$, more precisely the even Betti numbers of $\mathcal{X}_k$ are obtained as follows (the odd Betti numbers of $\mathcal{X}_k$ are all equal to 0).

Theorem 1. *[see [7]] The $2i$-th Betti number of $\mathcal{X}_k$ is given by*

$$\beta_{2i}(\mathcal{X}_k) = \#(\textit{permutations of } k+1 \textit{ different numbers in } [n] \textit{ with } i \textit{ descents})$$

These Betti numbers are a quite natural generalization of the Eulerian numbers. In fact we notice that when $k = n-2$ the formula above states that

$\beta_{2i}(\mathcal{X}_{n-2})$ is the number of permutations of $n-1$ different numbers in $[n]$ with i descents, which is equivalent to saying that it is the number of permutations of n numbers with i descents, i.e. the Eulerian number $A(n, i+1)$. On the geometrical side, we observe that since $\mathcal{X}_{n-2}$ is the permutohedral variety, it is well known that its Betti numbers $\beta_{2i}(\mathcal{X}_{n-2})$ coincide with the Eulerian numbers.

Let now k be an integer such that $0 \leq k \leq n-2$ and let us now consider in $\mathbb{C}^n$ the arrangement $\mathcal{G}_k$ given by all the coordinate subspaces of dimension $\leq k$. This arrangement is built according to the definition given by De Concini and Procesi in [2]; the space that we obtain by blowing up all the subspaces in $\mathcal{G}_k$ is a wonderful model Y_k whose cohomology ring is isomorphic to the cohomology of the corresponding projective model i.e. $\mathcal{X}_k$. In [4] a description of a monomial basis for the integer cohomology of De Concini–Procesi wonderful models is provided, and this basis for Y_k can be described in the following way.

A monomial in this basis is a product of Chern classes of (transforms of) subspaces in $\mathcal{G}_k$. Let $\zeta_{G_1}^{\alpha_1}\zeta_{G_2}^{\alpha_2}\cdots\zeta_{G_s}^{\alpha_s}$ be such a monomial. Then

(1) $G_1, G_2, \ldots, G_s$ is a chain in $\mathcal{G}_k$ with respect to inclusion (say $G_1 \subset \cdots \subset G_s$) where in each inclusion the codimension is greater than 1 and the codimension of G_s in $\mathbb{C}^n$ is greater than 1;

(2) for every i, α_i is a positive integer less than the codimension of G_i in G_{i+1} (for $i = s$ we consider the codimension of G_s in $\mathbb{C}^n$).

We now use for brevity a notation as in the following example: if G_1 is the subspace given by the equations $x_1 = x_2 = x_4 = x_6 = 0$ then we associate to G_1 the subset $1, 2, 4, 6$ of $[n]$, we denote $G_1 = H_{1,2,4,6}$ and we call $\zeta_{1,2,4,6}$ its associated Chern class. According to this notation, there is a bijection between the elements in $\mathcal{G}_k$ and the subsets of $[n]$ whose cardinality is greater than or equal to $n-k$.

Remark 2. As an example let $k = 7$ $n = 11$ and consider the monomial

$$\zeta^2_{\{1,2,4,5\}}\zeta^2_{\{1,2,4,5,6,7,8\}}\zeta_{\{1,2,4,5,6,7,8,9,11\}} \tag{4}$$

which is an element of the basis of the cohomology of $\mathcal{X}_7$.

For instance, notice that here the exponent of $\zeta_{\{1,2,4,5\}}$ is strictly less than 4, which is the codimension of $H_{1,2,4,5}$ in $\mathbb{C}^{11}$.

We now show an algorithm that produces a bijection between this monomial basis of $\mathcal{X}_k$ and the elements of $\mathcal{S}_n$ whose first hook from the right in the hook factorization has cardinality at least $n-k$ (see also [5, Section 5.2] for the case $k = n-2$). This bijection is grade-preserving provided that we consider in $\mathcal{S}_n$ the grade induced by the statistic lec. We illustrate the algorithm producing a permutation σ from the above mentioned monomial

$$\zeta^2_{\{1,2,4,5\}}\zeta^2_{\{1,2,4,5,6,7,8\}}\zeta_{\{1,2,4,5,6,7,8,9,11\}}$$

- We first look at the elements in $\{1,\ldots,11\}$ that do not appear in the description of the monomial. We write them in increasing order obtaining the non-hook part γ of σ. In our example we have $\{1,\ldots,11\} \setminus \{1,2,4,5,6,7,8,9,11\} = \{3,10\}$ so $\gamma = 3\ 10$.
- The first (from the right) hook of σ is obtained by taking the numbers 1,2,4,5, that belong to the smallest set of the chain, and by arranging them in such a way that the number of inversions is equal to the exponent 2 of $\zeta^2_{\{1,2,4,5\}}$. So we get 4125.
- The second (from the right) hook of σ is formed using the numbers of the second set of the chain, i.e. $\{1,2,4,5,6,7,8\}$, that do not appear in the smallest one. In the example those numbers are $\{6,7,8\}$ which is $\{1,2,4,5,6,7,8\} \setminus \{1,2,4,5\}$, arranged in such a way as to obtain 2 inversions since 2 is the exponent of $\zeta_{\{1,2,4,5,6,7,8\}}$ in the monomial: 867.
- Analogously, the last (from the right) hook is 11 9

In the end we obtain $\sigma = 3\ 10\ 11\ 98674125$, which satisfies $\mathrm{lec}(\sigma) = 5$.

From this bijection we deduce another way to obtain the Betti numbers of $\mathcal{X}_k$:

Theorem 2. *Let $0 \le k \le n-2$ and $i > 0$. The $2i$-th Betti number of $\mathcal{X}_k$ is given by*

$$\beta_{2i}(\mathcal{X}_k) = \#(\textit{elements } \sigma \in \mathcal{S}_n, \textit{ such that } \mathrm{lec}(\sigma) = i, \textit{ whose first hook from the right has cardinality at least } n-k)$$

Therefore considering two different cohomology bases of $\mathcal{X}_k$ allows us to deduce, by double counting, the following result:

Corollary 1. *Let $0 \le k \le n-1$ and $i \ge 0$. The cardinality of the set of the permutations of $k+1$ different numbers in $[n]$ with i descents is equal to the cardinality of the set of the elements $\sigma \in \mathcal{S}_n$, such that $\mathrm{lec}(\sigma) = i$, whose first hook from the right has cardinality at least $n-k$.*

Remark 3. The statement of the Corollary 1 includes also the cases $i = 0$ and $n = k-1$. Namely if $i = 0$ we have only one permutation of $k+1$ different numbers in $[n]$ with no descents, and only one permutation $\sigma \in \mathcal{S}_n$ such that $\mathrm{lec}(\sigma) = 0$, that is the identity.

If $k = n-1$ the statement immediately follows from the case $k = n-2$. Indeed, since any hook has cardinality at least 2 by definition, the requests that the first hook from the right has cardinality at least 1 or 2 are equivalent, and as we have already observed, the number of permutations of $n-1$ different numbers in $[n]$ with i descents is equal to the number of permutations of n numbers with i descents.

3 An explicit bijection of $\mathcal{S}_n$ preserving des-lec statistic: reverse waves

Motivated by Corollary 1 we now search for a combinatorial bijection between the set of the permutations of $k+1$ different numbers in $[n]$ with i descents and the set of the elements $\sigma \in \mathcal{S}_n$, such that $\mathrm{lec}(\sigma) = i$, whose first hook from the right has cardinality at least $n-k$. In this section we focus on the case $k = n-2$, which as already observed is equivalent to the case $k = n-1$; this means that we search for a combinatorial bijection of the symmetric group $\mathcal{S}_n$ that sends the des statistic to the lec statistic.

To do so, we will present an explicit bijection of the set

$$W = \{\, a_1 \cdots a_\ell \mid \ell \in \mathbb{N},\, a_i \in \mathbb{N}^* \,\forall i \text{ and } a_i \neq a_j \,\forall\, i \neq j \,\}$$

of all finite words with distinct entries, such that if we restrict to the symmetric group $\mathcal{S}_n$, it becomes a bijection of $\mathcal{S}_n$.

Remark 4. We can evaluate the lec of a word w in a recursive way, as suggested by the Remark 1. If $w = \gamma * \alpha_1 * \cdots * \alpha_k$ is the Hook factorization, and $k = 0$, then $\mathrm{lec}(w) = \mathrm{lec}(\gamma) = 0$, while if $k > 0$, then we have by definition that

$$\mathrm{lec}(w) = \mathrm{lec}(\gamma * \alpha_1 * \cdots * \alpha_{k-1}) + \mathrm{lec}(\alpha_k). \tag{5}$$

If a hook α has length ℓ, then we obtain that $1 \leq \mathrm{lec}(\alpha) \leq \ell - 1$. Moreover, if we have ℓ distinct numbers $a_1 < a_2 < \cdots < a_\ell$, for any $0 \leq d \leq \ell - 1$, we have exactly one way to rearrange them to create a hook, possibly trivial if $d = 0$, α_d with $\mathrm{lec}(\alpha_d)$ equal to d, that is, $\alpha_d = a_{d+1} a_1 a_2 \cdots a_d a_{d+2} \cdots a_\ell$.

We will define a similar way of evaluating descents by decomposition in smaller words.

Definition 2. Let $w = w_1 \cdots w_\ell \in W$ be a non-empty word of length ℓ. Let c be the position in w of the maximum element of w, we say that w is a *reverse wave* if $c \neq \ell$ and

$$w_1 < w_2 < \cdots < w_{c-1} < w_\ell < w_{\ell-1} < \cdots < w_c. \tag{6}$$

We will also denote a word with elements in increasing order, that is the case with $c = \ell$, as a *trivial reverse wave.*

Remark 5. If w is a reverse wave, then by definition, any prefix of w is also a reverse wave, possibly trivial.

Remark 6. Similarly to what was observed in Remark 4, if a reverse wave rw has length ℓ, then we have that $1 \leq \mathrm{des}(rw) \leq \ell - 1$. Moreover if we have ℓ distinct numbers $b_1 < b_2 < \cdots < b_\ell$, for any $0 \leq d \leq \ell - 1$, we have exactly one way to rearrange them to create a reverse wave, possibly trivial if $d = 0$, r_d with $\mathrm{des}(rw_d)$ equal to d, that is, $rw_d = b_1 b_2 \cdots b_{\ell-d-1} b_\ell b_{\ell-1} \cdots b_{\ell-d}$.

Definition 3. Let $w = w_1 \cdots w_\ell \in W$ be a non-empty word of length ℓ. The *special reverse wave* of w is the prefix $\mathrm{srw}(w) = w_1 w_2 \cdots w_t \cdots w_{s-1} w_s$ of w where:

- $1 \leq t < \ell$ is the index of the first descent of w, so it is the minimum such that $w_t > w_{t+1}$. If w has no descents, if and only if elements of w are in increasing order, then $\mathrm{srw}(w) = w$.

- $t \leq s \leq \ell$ is the maximum such that $w_t > w_{t+1} > \cdots > w_{s-1} > w_s > w_{t-1}$, where, if $t = 1$, we consider $w_0 = 0$. It exists since $w_t > w_{t-1}$, by minimality of t, and if $t = 1$, then is it true that $w_t > 0 = w_{t-1}$ by definition of W.

Remark 7. If $\mathrm{srw}(w) = w_1 \cdots w_t \cdots w_{s-1} w_s$ is the special reverse wave of w, then we have

$$w_1 < w_2 < \cdots < w_{t-1} < w_s < w_{s-1} < \cdots < w_t \tag{7}$$

and in particular that $\mathrm{srw}(w)$ is a reverse wave, possibly trivial. This is the maximal prefix of σ that is also a reverse wave. If w has no descents, then $\mathrm{srw}(w) = w$, and in this case $\mathrm{srw}(w)$ is a trivial reverse wave.

We also have, if $s \neq \ell$, that $w_{s+1} > w_s$ or that $w_{s+1} < w_{t-1} < w_s$.

We will denote by $w \setminus \mathrm{srw}(w) = w_{s+1} \cdots w_\ell$ the suffix obtained by removing from w its special reverse wave. If $\mathrm{srw}(w) = w$ then $w \setminus \mathrm{srw}(w)$ is the empty word.

Example 4. Let $\sigma = 45762318 \in \mathcal{S}_8 \subset W$, then its special reverse wave is $\mathrm{srw}(\sigma) = 4576$ and $w \setminus \mathrm{srw}(w) = 2318$.

We introduce a new statistic that will allow us to correlate the number of the descents of a word with those of its special reverse wave.

Definition 4. Let $w = w_1 \cdots w_\ell \in W$ be a non-empty word and $\mathrm{srw}(w) = w_1 w_2 \cdots w_t \cdots w_{s-1} w_s$ be its special reverse wave. We denote *the number of special reverse descents* of w by

$$\mathrm{srdes}(w) = \mathrm{des}(\mathrm{srw}(w)) + \chi(w_s > w_{s+1}) \tag{8}$$

where $\chi(P)$ equals 1 if the property P is true, and 0 otherwise, and if $s = \ell$ then $\chi(w_s > w_{s+1}) = 0$.

Remark 8. We have that $\mathrm{des}(w) = \mathrm{srdes}(w) + \mathrm{des}(w \setminus \mathrm{srw}(w))$. Indeed if $s = \ell$ we obtain that $w = \mathrm{srw}(w)$ and $\mathrm{srdes}(w) = \mathrm{des}(\mathrm{srw}(w)) = \mathrm{des}(w)$ as wanted, while if $s < \ell$, we get that

$$\begin{aligned}\operatorname{des}(w) = \operatorname{des}(w_1 \cdots w_\ell) &= \operatorname{des}(w_1 \cdots w_s) + \chi(w_s > w_{s+1}) + \operatorname{des}(w_{s+1} \cdots w_l) \\ &= \operatorname{srdes}(w) + \operatorname{des}(w \setminus \operatorname{sw}(w)).\end{aligned} \tag{9}$$

Example 5. Let $\sigma = 45762318 \in \mathcal{S}_8 \subset W$, as in the previous example. We have that $\operatorname{srdes}(w) = \operatorname{des}(4576) + \chi(6 > 2) = 1 + 1 = 2$ and $\operatorname{des}(w) = 3 = 2 + 1 = \operatorname{srdes}(w) + \operatorname{des}(2318)$.

Proposition 1. *Let $w = w_1 \cdots w_\ell \in W$ be a non increasing word and $\operatorname{sw}(w) = w_1 \cdots w_s$ its special reverse wave. Then we have that*

$$1 \leq \operatorname{srdes}(w) \leq s - 1. \tag{10}$$

Moreover if w is an increasing word then $\operatorname{srdes}(w) = 0$.

Proof. First of all let us prove that $\operatorname{srdes}(w) \geq 1$ if w is not increasing. If $\operatorname{srdes}(w) = 0$ then $\operatorname{des}(\operatorname{srw}(w)) = 0$ and so the first descent of w is exactly the one between the last element of $\operatorname{srw}(w)$ and the first of $w \setminus \operatorname{srw}(w)$. This implies that $\operatorname{srdes}(w) = 1$, that is an absurd.

Let us now demonstrate the other inequality. We have that $\operatorname{des}(\operatorname{srw}(w)) \leq s - 1$ and so $\operatorname{srdes}(w) = \operatorname{des}(\operatorname{srw}(w)) + \chi(w_s > w_{s+1}) \leq s$ and the equality holds if and only if $\operatorname{srw}(w)$ is a decreasing word and $w_s > w_{s+1}$. This is an absurd, since if $\operatorname{srw}(w)$ is a decreasing word and w_{s+1} is not in the special reverse wave of w, then $w_s < w_{s+1}$.

If w is an increasing word then $\operatorname{srw}(w) = w$ and $\operatorname{srdes}(w) = \operatorname{des}(w) = 0$ as wanted. $\boxed{QED}$

Proposition 2. *Let $a = a_1 \cdots a_p$ and $b = b_1 \cdots b_q$ two words in W with distinct entries and $1 \leq d \leq q - 1$. Then there exists a unique word* $\operatorname{rins}(a, b, d)$ *such that* $\operatorname{srw}(\operatorname{rins}(a, b, d))$ *is a rearrangement of the elements of b and*

$$\operatorname{rins}(a, b, d) \setminus \operatorname{srw}(\operatorname{rins}(a, b, d)) = a, \tag{11}$$

$$\operatorname{srdes}(\operatorname{rins}(a, b, d)) = d. \tag{12}$$

Moreover for all $w \in W$ that are not increasing,

$$\operatorname{rins}(w \setminus \operatorname{srw}(w), \operatorname{srw}(w), \operatorname{srdes}(w)) = w. \tag{13}$$

Proof. Let $B = \{\beta_1 < \cdots < \beta_q\}$ the set of the elements of b. We will prove that $\mathsf{r}(a, b, d)$, that is equal to $\operatorname{rwave}(b, d) * a = \beta_1 \cdots \beta_{q-d-1}\beta_q \cdots \beta_{q-d} a_1 \cdots a_p$ if $\beta_{q-d} < a_1$ or if $p = 0$ and to $\operatorname{rwave}(b, d-1) * a = \beta_1 \cdots \beta_{q-d}\beta_q \cdots \beta_{q-d+1} a_1 \cdots a_p$

else, is the only word that satisfies the required properties, where $\mathrm{rwave}(b, x)$ is the only reverse wave with the same elements of b and $0 \leq x \leq q-1$ descents (see Remark 6).

We have that both $\mathrm{rwave}(b, d)$ and $\mathrm{rwave}(b, d-1)$ are reverse waves of length q, so the special reverse wave of $\mathsf{r}(a, b, d)$ has length at least q. Moreover in $\mathrm{rwave}(b, d)$ there is at least one descent, since $d \geq 1$, and if $\beta_{q-d} < a_1$ or $p = 0$, then $\mathsf{r}(a, b, d) = \mathrm{rwave}(b, d)$. In these cases it is clear that $\mathsf{r}(a, b, d) \setminus \mathrm{srw}(\mathsf{r}(a, b, d)) = a$ and that

$$\mathrm{srdes}(\mathsf{r}(a, b, d)) = \mathrm{des}(\mathrm{rwave}(b, d)) + \chi(\beta_{l-d} > a_1) = d + 0 = d.$$

If $p \geq 1$ and $\beta_{q-d} > a_1$ then the position of the first descent of $\mathsf{r}(a, b, d)$ is $q - d + 1$ and since $\beta_{q-d} > a_1$, then $\mathsf{r}(a, b, d) = \mathrm{rwave}(b, d-1)$. In this case it is clear that $\mathsf{r}(a, b, d) \setminus \mathrm{srw}(\mathsf{r}(a, b, d)) = a$ and that

$$\mathrm{srdes}(\mathsf{r}(a, b, d)) = \mathrm{des}(\mathrm{rwave}(b, d-1)) + \chi(\beta_{l-d} > a_1) = d - 1 + 1 = d.$$

So in every case $\mathsf{r}(a, b, d)$ satisfies the properties of the proposition. Let us now demonstrate that it is the only possible word, and so that $\mathrm{rins}(a, b, d) = \mathsf{r}(a, b, d)$.

Since $\mathrm{srw}(\mathrm{rins}(a, b, d))$ is a rearrangement of the elements of b, we obtain that

$$\mathrm{srw}(\mathrm{rins}(a, b, d)) = \mathrm{rwave}(b, y) = \beta_1 \cdots \beta_{q-y-1} \beta_q \cdots \beta_{q-y}$$

with a proper $0 \leq y \leq q-1$, and thanks to equation (11) that $\mathrm{rins}(a, b, d) = \mathrm{rwave}(b, y) * a$. Moreover, if $p = 0$ equation (12) implies that

$$d = \mathrm{srdes}(\mathrm{rins}(a, b, d)) = \mathrm{des}(\mathrm{srw}(\mathrm{rins}(a, b, d))) = \mathrm{des}(\mathrm{rwave}(b, y)) = y$$

and so that $\mathrm{rins}(a, b, d) = \mathsf{r}(a, b, d)$.

If $p > 0$, equation (12) implies that

$$\begin{aligned} d &= \mathrm{srdes}(\mathrm{rins}(a, b, d)) = \mathrm{des}(\mathrm{srw}(\mathrm{rins}(a, b, d))) \\ &= \mathrm{des}(\mathrm{rwave}(b, y)) + \chi(\beta_{q-y} > a_1) = y + \chi(\beta_{q-y} > a_1), \end{aligned}$$

and so that $y = d - 1$ or $y = d$.

If $y = d - 1$, then the position of the first descent in $\mathrm{rins}(a, b, d)$ is $q - y = q - d + 1$, and since a_1 is not in the special reverse wave of $\mathrm{rins}(a, b, d)$, we have that $a_1 > \beta_{q-d}$, and so that $\mathrm{rins}(a, b, d) = \mathsf{r}(a, b, d)$.

If $y = d$, then we have that $\chi(\beta_{q-y} > a_1) = \chi(\beta_{q-d} > a_1) = 0$ and so that $\beta_{q-d} < a_1$. Then, also in this case, $\mathrm{rins}(a, b, d) = \mathsf{r}(a, b, d)$ as wanted.

To prove the identity of the equation (13) we observe that $1 \leq \mathrm{srdes}(w) \leq \ell(\mathrm{srw}(w)) - 1$ by the Proposition 1 and so that $\mathrm{rins}(w \setminus \mathrm{srw}(w), \mathrm{srw}(w), \mathrm{srdes}(w))$

is well-defined. Moreover w is a word that satisfies the requested properties, and thanks to the just proved uniqueness of rins, we have that

$$\mathrm{rins}(w \setminus \mathrm{srw}(w), \mathrm{srw}(w), \mathrm{srdes}(w)) = w.$$

$\boxed{QED}$

Now we can define the desired bijection.
Let $\theta : W \to W$ be the map such that:

- If $w = \varepsilon$ is the empty word then $\theta(\varepsilon) = \varepsilon$.
- Otherwise
$$\theta(w) = \theta(w \setminus \mathrm{srw}(w)) * \delta_d(\mathrm{srw}(w)) \tag{14}$$
where the $*$ operator is the concatenation of words and $\delta_d(\mathrm{srw}(w)) = c_{d+1}c_1c_2\cdots c_dc_{d+2}\cdots c_m$ where $c_1 < c_2 < \cdots < c_m$ are the elements of $\mathrm{srw}(w)$ and $d = \mathrm{srdes}(w)$, is the unique rearrangement of $\mathrm{srw}(w)$ that is a hook with lec equal to d.

Theorem 3. *The map $\theta : W \to W$ just defined is a well defined bijection of W, such that*
$$\mathrm{des}(w) = \mathrm{lec}(\theta(w)). \tag{15}$$
Moreover the restriction to the symmetric group $\theta_{\mathcal{S}_n} : \mathcal{S}_n \to \mathcal{S}_n$ is a bijection.

Proof. We will prove by induction on the length ℓ of $w \in W$ that $\theta(w)$ is a rearrangement of w, and that it is a well defined map. If $\ell = 0$ then $w = \varepsilon$ is the empty word and by definition $\theta(\varepsilon) = \varepsilon$, that is fine. If w has length $\ell > 0$, then by definition $w \setminus \mathrm{srw}(w)$ is a word of length strictly less than ℓ, and so by the inductive hypothesis $\theta(w \setminus \mathrm{srw}(w))$ is a rearrangement of $w \setminus \mathrm{srw}(w)$. Moreover $\delta_d(\mathrm{srw}(w))$ is defined as a rearrangement of $\mathrm{srw}(w)$, so this means that $\theta(w)$ is a rearrangement of w, and θ is a well defined map.

To prove that θ is a bijection we will define its inverse η as the map such that:

- If $w = \varepsilon$ is the empty word then $\eta(\varepsilon) = \varepsilon$.
- Otherwise, if the rightmost hook $\mathrm{h}(w)$ of w is not trivial
$$\eta(w) = \mathrm{rins}(\eta(w \setminus \mathrm{h}(w)), \mathrm{h}(w), \mathrm{lec}(\mathrm{h}(w))), \tag{16}$$
where $w \setminus \mathrm{h}(w)$ is the prefix of w obtained removing $\mathrm{h}(w)$. If $\mathrm{h}(w)$ is a trivial hook, then $\mathrm{h}(w) = w$ and in this case $\eta(w) = w$

We can notice that from the definition of θ we have that if w is increasing then $\mathrm{srw}(w) = w$ and $\theta(w) = w$, and so it is clear that θ and η are inverse when restricted to the subset of increasing words, because they are both the identity in this case.

Now let us consider the case where $w \in W$ is not increasing.

Since $w \setminus \mathrm{h}(w)$ has length strictly less than w and, by definition the special reverse wave of $\eta(w)$ is a rearrangement of $\mathrm{h}(w)$, using inductive reasoning, entirely analogous to what was done previously for θ, we obtain that $\eta(w)$ is a rearrangement of w and that it is a well defined map.

Let us prove that they are inverse maps by induction on the length ℓ of w. If $\ell = 0$ it is clear. If $\ell > 0$ then

$$\begin{aligned} \theta(\eta(w)) &= \theta(\mathrm{rins}(\eta(w \setminus \mathrm{h}(w)), \mathrm{h}(w), \mathrm{lec}(\mathrm{h}(w)))) \\ &= \theta(\eta(w \setminus \mathrm{h}(w))) * \delta_{\mathrm{lec}(\mathrm{h}(w))}(\mathrm{h}(w)) \\ &= (w \setminus \mathrm{h}(w)) * \mathrm{h}(w) = w, \end{aligned} \tag{17}$$

where the second equality is true thanks to Proposition 2 and because the special reverse wave of $\mathrm{rins}(\eta(w \setminus \mathrm{h}(w)), \mathrm{h}(w), \mathrm{lec}(\mathrm{h}(w)))$ is a rearrangement of $\mathrm{h}(w)$. The last equality is true thanks to inductive hypothesis, since $w \setminus \mathrm{h}(w)$ has length strictly less than ℓ, and thanks to the fact that the unique rearrangement of the elements of $\mathrm{h}(w)$ that is a hook with lec equal to $\mathrm{lec}(\mathrm{h}(w))$ is $\mathrm{h}(w)$ itself. Moreover

$$\begin{aligned} \eta(\theta(w)) &= \mathrm{rins}(\eta(\theta(w) \setminus \mathrm{h}(\theta(w))), \mathrm{h}(\theta(w)), \mathrm{lec}(\mathrm{h}(\theta(w)))) \\ &= \mathrm{rins}(\eta(\theta(w) \setminus \delta_{\mathrm{srdes}(w)}(\mathrm{srw}(w))), \delta_{\mathrm{srdes}(w)}(\mathrm{srw}(w)), \mathrm{srdes}(w)) \\ &= \mathrm{rins}(\eta(\theta(w \setminus \mathrm{srw}(w))), \mathrm{srw}(w), \mathrm{srdes}(w)) \\ &= \mathrm{rins}(w \setminus \mathrm{srw}(w)), \mathrm{srw}(w), \mathrm{srdes}(w)) = w, \end{aligned} \tag{18}$$

where the second equality is true since the rightmost hook of $\theta(w)$ is by definition $\delta_{\mathrm{srdes}(w)}(\mathrm{srw}(\theta(w)))$ and $\mathrm{lec}(\mathrm{h}(\theta(w))) = \mathrm{srdes}(w)$, the third equality is true since $\delta_{\mathrm{srdes}(w)}(\mathrm{srw}(w))$ is a rearrangement of $\mathrm{srw}(w)$ and by the definition of rins and thanks to the definition of $\theta(w)$. The fourth equality holds by inductive hypothesis, since the length of $w \setminus \mathrm{srw}(w)$ is strictly less than ℓ and the last is true thanks to Proposition 2. We have proved that θ and η are inverse, and so that they are bijections of W. Moreover we have already proved that they permute the elements of the words, so that their restrictions to the symmetric group $\mathcal{S}_n$ are two bijections.

We can also show the identity of equation (15) by an inductive argument on the length ℓ of $w \in W$. If $\ell = 0$, then $w = \varepsilon$ is the empty word and

$\mathrm{lec}(\theta(\varepsilon)) = \mathrm{lec}(\varepsilon) = 0 = \mathrm{des}(\varepsilon$. If w has length $\ell > 0$ we have that

$$\begin{aligned}\mathrm{lec}(\theta(w)) &= \mathrm{lec}(\theta(w \setminus \mathrm{srw}(w))) + \mathrm{lec}(\delta_{\mathrm{srdes}(w)}(\mathrm{srw}(w)) \\ &= \mathrm{des}(w \setminus \mathrm{srw}(w)) + \mathrm{srdes}(w) = \mathrm{des}(w),\end{aligned} \tag{19}$$

where the first equality holds since $\delta(\mathrm{srw}(w))$ is the rightmost hook of $\theta(w)$, the second is true by inductive hypothesis and by definition of $\delta_{\mathrm{srdes}(w)}$. The last equality is true thanks to Remark 8. $\boxed{QED}$

Example 6. Let $\sigma = 45762318 \in \mathcal{S}_8 \subset W$, then

$$\begin{aligned}\theta(45762318) &= \theta(2318) * \delta_2(4576) = \theta(18) * \delta_1(23) * 6457 \\ &= \theta(\varepsilon) + \delta_0(18) * 32 * 6457 = \varepsilon * 18 * 32 * 6457 \\ &= 18326457\end{aligned}$$

and

$$\begin{aligned}\eta(18326457) &= \mathrm{rins}(\eta(1832), 6457, 2) = \mathrm{rins}(\mathrm{rins}(\eta(18), 32, 1), 6457, 2) \\ &= \mathrm{rins}(\mathrm{rins}(18, 32, 1), 6457, 2) = \mathrm{rins}(2318, 6457, 2) = 45762318.\end{aligned}$$

Moreover $\mathrm{des}(45762318) = 3 = \mathrm{lec}(18326457) = \mathrm{lec}(\theta(45762318))$.

Remark 9. If $w = \gamma * \alpha_1 * \cdots * \alpha_k$ is the Hook factorization of a word $w \in W$, then we can write

$$w = \mathrm{conc}(\cdots \mathrm{conc}(\mathrm{conc}(\gamma, \alpha_1), \alpha_2) \cdots, \alpha_k),$$

where conc is the binary operation that concatenates words, i.e. $\mathrm{conc}(w', w'') = w' * w''$ for all $w', w'' \in W$.

We have that the map η is nothing more than the map obtained by replacing each conc with rins, namely:

$$\eta(w) = \mathrm{rins}(\cdots \mathrm{rins}(\mathrm{rins}(\gamma, \alpha_1, \mathrm{lec}(\alpha_1)), \alpha_2, \mathrm{lec}(\alpha_2)) \cdots, \alpha_k, \mathrm{lec}(\alpha_k)).$$

Theorem 3 therefore also implies that for every word in W, it can be obtained, in a unique manner, through an ordered and finite sequence of r-insertions of reverse waves, starting from an increasing word. The map θ will then be analogously the one that replaces reverse waves β_i with the corresponding hooks $\delta(\beta_i)$ and r-insertion operations with concatenation operations.

In particular, the set of permutations $\sigma \in \mathcal{S}_n$ with $\mathrm{lec}(\sigma) = i$ and whose rightmost hook has length at least $n - k$ is in bijection, via η and θ, with the set of permutations $\sigma \in \mathcal{S}_n$ with $\mathrm{des}(\sigma) = i$ and whose reverse special wave has length at least $n - k$.

4 An explicit bijection in the general case of permutations of $k+1$ different numbers

In this section we will show an explicit combinatorial bijection between the set of permutations of $k+1$ different numbers in $[n]$ (with $0 \leq k \leq n-2$), with i descents and the subset of $\mathcal{S}_n$ made of the permutations σ such that $\operatorname{lec}(\sigma) = i$ and whose first hook from the right has cardinality at least $n-k$, proving the Corollary 1 in a combinatorial way.

To do this we will construct a bijection that preserves the number of descents between the set of permutations of $k+1$ different numbers in $[n]$ and the subset of $\mathcal{S}_n$ of permutations with special reverse wave of length at least $n-k$. Thanks to the Theorem 3 and to the Remark 9, composing this bijection with the map θ of the previous section, gives us the desired bijection.

The first step is the construction of a descent preserving bijection between $\mathcal{S}_{n,k+1}$ and a proper subset of $\mathcal{S}_n$

Theorem 4. *The map $\mu : \mathcal{S}_{n,k+1} \to \{\, \sigma \in \mathcal{S}_n \mid \ell(\operatorname{srw}(\sigma)) \geq n-k \,\}$ such that for every $(\alpha_0, \rho) \in \mathcal{S}_{n,k+1}$*

$$\mu((\alpha_0, \rho)) = \alpha_1\alpha_2 \cdots \alpha_{n-k-m-1}\alpha_{n-k-1}\alpha_{n-k-2} \cdots \alpha_{n-k-m}\rho_1 \cdots \rho_{k+1} \tag{20}$$

where $(\alpha_0, \rho) = (\{\, \alpha_1 < \cdots < \alpha_{n-k-1} \,\}, \rho_1 \cdots \rho_{k+1})$ and $m = \#\{\, \alpha \in \alpha_0 \mid \alpha > \rho_1 \,\}$, is a bijection that preserves the number of descents.

Its inverse is the map $\nu : \{\, \sigma \in \mathcal{S}_n \mid \ell(\operatorname{srw}(\sigma)) \geq n-k \,\} \to \mathcal{S}_{n,k+1}$ such that for every $\sigma = \sigma_1 \cdots \sigma_n$

$$\nu(\sigma) = (\{\, \sigma_1, \ldots, \sigma_{n-k-1} \,\}, \sigma_{n-k} \cdots \sigma_n). \tag{21}$$

Proof. If we call $\sigma = \mu((\alpha_0, \rho))$, by definition of the map μ we have that

$$\alpha_1\alpha_2 \cdots \alpha_{n-k-m-1}\alpha_{n-k-1}\alpha_{n-k-2} \cdots \alpha_{n-k-m}\rho_1$$

is a prefix of σ, of length $n-k$, that is also a reverse wave. By Remark 3 $\operatorname{srw}(\sigma)$ is the maximal prefix of σ that is a reverse wave and so we have that $\operatorname{srw}(\sigma)$ has length at least $n-k$. This implies that the map μ is well-defined.

In particular the map μ is the concatenation of the unique reverse wave with elements in $\alpha_0 \sqcup \{\, \rho_1 \,\}$ and m descents with $\rho_2 \cdots \rho_{k+1}$.

The number of descents of (α_0, ρ) is by definition

$$\operatorname{des}((\alpha_0, \rho)) = m + \operatorname{des}(\rho) = \operatorname{des}(\mu((\alpha_0, \rho)))$$

and so we have that μ preserves the number of descents.

Let us demonstrate that the map ν is the inverse of μ. We have that

$$\begin{aligned}\nu(\mu((\alpha_0,\rho))) &= \nu(\alpha_1\alpha_2\cdots\alpha_{n-k-m-1}\alpha_{n-k-1}\alpha_{n-k-2}\cdots\alpha_{n-k-m}\rho_1\cdots\rho_{k+1}) \\ &= (\{\,\alpha_1,\ldots,\alpha_{n-k-1}\,\},\rho_1\cdots\rho_{k+1}) = (\alpha_0,\rho).\end{aligned}$$

Moreover if $\sigma = \sigma_1\cdots\sigma_n \in \{\,\sigma \in \mathcal{S}_n \mid \ell(\mathrm{srw}(\sigma)) \geq n-k\,\}$ and $\mathrm{srw}(\sigma) = \sigma_1\cdots\sigma_s$, with $s \geq n-k$, we have that $\sigma_1\cdots\sigma_{n-k-1}\sigma_{n-k}$ is a prefix of a reverse wave, and so, thanks to Remarks 5 and 6, it is the unique reverse wave, with exactly $y = \#\{\,1 \leq i \leq n-k-1 \mid \sigma_i > \sigma_{n-k}\,\}$ descents. Therefore we have that

$$\begin{aligned}\mu(\nu(\sigma)) &= \mu((\{\,\sigma_1,\ldots,\sigma_{n-k-1}\,\},\sigma_{n-k}\sigma_{n-k+1}\cdots\sigma_n)) \\ &= \sigma_1\cdots\sigma_{n-k-1}\sigma_{n-k} * \sigma_{n-k+1}\cdots\sigma_n = \sigma,\end{aligned}$$

thanks to the previous characterization of μ. $\boxed{QED}$

Example 7. Let $(\alpha_0,\tau) = (\{\,4,5,7\,\},62318) \in \mathcal{S}_{8,5}$, then we have that $m = \{\,\alpha \in \{\,4,5,7\,\} \mid \alpha > 6\,\} = 1$ and so $\mu((\{\,4,5,7\,\},62318)) = 45762318$.

Clearly we have that $\nu(45762318) = (\{\,4,5,7\,\},62318)$. Moreover $\mathrm{des}((\{\,4,5,7\,\},62318)) = 1+2 = 3 = \mathrm{des}(45762318)$ and $\ell(\mathrm{srw}(45762318)) = \ell(4576) = 4 \geq 4 = 8-4$.

We have proved, thanks to Remark 9, Theorems 3 and 4 that the map $\theta \circ \mu : \mathcal{S}_{n,k+1} \to \{\,\sigma \in \mathcal{S}_n \mid \ell(\mathrm{h}(\sigma)) \geq n-k\,\}$ is a bijection such that, for every $(\alpha_0,\rho) \in \mathcal{S}_{n,k+1}$ we have that $\mathrm{des}((\alpha_0,\rho)) = \mathrm{lec}(\theta(\mu((\alpha_0,\rho))))$.

This provides another proof of Corollary 1.

5 Second bijection (Foata–Han) preserving des-lec statistic: waves

In this section we present a different bijection ψ of the symmetric group $\mathcal{S}_n$ that sends the des statistic to the lec statistic.

Up to composition with trivial bijections, this bijection ψ appears to coincide with the map F defined by Foata and Han in [3, Section 6]. In particular, it seems to hold that:

$$\psi = \mathbf{c} \circ F \circ \mathbf{c} \circ \mathbf{r}, \tag{22}$$

where $\mathbf{c}(\sigma) = (n+1-\sigma_1)\ldots(n+1-\sigma_n)$ and $\mathbf{r}(\sigma) = \sigma_n\ldots\sigma_1$. We have verified this equality computationally for n small. We present it anyway because it can be constructed explicitly and directly, using the methods described in the previous sections.

To do so, we will present an explicit bijection of the set

$$W = \{\,a_1\cdots a_\ell \mid \ell \in \mathbb{N},\, a_i \in \mathbb{N}^*\,\forall i \text{ and } a_i \neq a_j\,\forall\, i \neq j\,\}$$

of all finite words with distinct entries, such that if we restrict to the symmetric group $\mathcal{S}_n$, it becomes a bijection of $\mathcal{S}_n$.

We will define a new way of evaluating descents by decomposition in smaller words.

Definition 5. Let $w = w_1 \cdots w_\ell \in W$ be a non-empty word of length ℓ. Let c be the position in w of the maximum element of w, we say that w is a *wave* if $c \neq \ell$ and

$$w_\ell < w_{\ell-1} < \cdots < w_{c+1} < w_1 < w_2 < \cdots < w_c. \tag{23}$$

We will also denote a word with elements in increasing order, that is the case with $c = \ell$, as a *trivial wave*.

Remark 10. If w is a wave of length ℓ and c is the position of its maximum, then $\mathrm{des}(w) = \ell - c$.

Definition 6. Let $w = w_1 \cdots w_\ell \in W$ be a non-empty word of length ℓ. The *special wave* of w is the subword $\mathrm{sw}(w) = w_r w_{r+1} \cdots w_t \cdots w_{s-1} w_s$ of w where:

- $1 \leq t < \ell$ is the index of the first descent of w, so it is the minimum such that $w_t > w_{t+1}$. If w has no descents, if and only if the elements of w are in increasing order, then $\mathrm{sw}(w) = w$.
- $1 \leq r \leq t$ is the minimum such that $w_r > w_{t+1}$. It exists since $w_t > w_{t+1}$.
- $t < s \leq \ell$ is the maximum such that $w_{t+1} > w_{t+2} > \cdots > w_{s-1} > w_s > w_{r-1}$, where, if $r = 1$, we consider $w_0 = 0$. It exists since $w_{t+1} > w_{r-1}$, by minimality of r, and if $r = 1$, then is it true that $w_{t+1} > 0 = w_{r-1}$ by definition of W.

Remark 11. If $\mathrm{sw}(w) = w_r w_{r+1} \cdots w_t \cdots w_{s-1} w_s$ is the special wave of w, then we have

$$w_1 < w_2 < \cdots < w_{r-1} < w_s < w_{s-1} < \cdots < w_{t+1} < w_r < \cdots < w_t \tag{24}$$

and in particular that $\mathrm{sw}(w)$ is a wave, since

$$w_s < w_{s-1} < \cdots < w_{t+1} < w_r < \cdots < w_t. \tag{25}$$

We also have, if $s \neq \ell$, that $w_{s+1} > w_s$ or $w_{s+1} < w_{r-1}$.

If w has no descents, then $\mathrm{sw}(w) = w$, and in this case $\mathrm{sw}(w)$ is a trivial wave.

We will denote by $w \setminus \mathrm{sw}(w) = w_1 \cdots w_{r-1} w_{s+1} \cdots w_\ell$ the word obtained by removing from w its special wave. If $\mathrm{sw}(w) = w$ then $w \setminus \mathrm{sw}(w)$ is the empty word.

Example 8. Let $\sigma = 13675428 \in \mathcal{S}_8 \subset W$, then its special wave is $\mathrm{sw}(\sigma) = 6754$ and $w \setminus \mathrm{sw}(w) = 1328$. From this example we can notice that the special wave is not necessarily maximal, indeed 67542 is also a wave, but $2 < 3$.

We can define a map that reverses the operation of removing the special reverse wave.

Proposition 3. *Let $a = a_1 \cdots a_p$ and $b = b_1 \cdots b_q$ two words in W with distinct entries and such that b is a non trivial wave. Let $0 \leq x \leq p$ be the maximum such that $a_1 < a_2 < \cdots < a_x < b_q$, where $x = 0$ if $a_1 > b_q$. Then the map* $\mathrm{ins}(a, b) = a_1 \cdots a_x b_1 \cdots b_q a_{x+1} \cdots a_p$ *satisfies*

$$\mathrm{sw}(\mathrm{ins}(a, b)) = b, \tag{26}$$

$$\mathrm{ins}(a, b) \setminus \mathrm{sw}(\mathrm{ins}(a, b)) = a, \tag{27}$$

and, for all $w \in W$ that are not increasing,

$$\mathrm{ins}(w \setminus \mathrm{sw}(w), \mathrm{sw}(w)) = w. \tag{28}$$

Proof. We observe that, since b is a wave and by the definition of x, we have that $b_i > b_q > a_x > \cdots > a_1$, for every $1 \leq i < q$. Moreover we have by maximality of x that $a_{x+1} > b_q$, or $x = p$. This implies that the first descent of $\mathrm{ins}(a, b)$ is the first descent of b, and that the extremal points of its special wave are exactly b_1 and b_p, and so that $\mathrm{sw}(\mathrm{ins}(a, b)) = b$. The equation (27) is clear from the definition of ins and from equation (26).

To prove the third property, let $w = w_1 \cdots w_\ell \in W$ be a word and $\mathrm{sw}(w) = w_r w_{r+1} \cdots w_t \cdots w_{s-1} w_s$ its special wave, with t as in the Definition 6. From equation (24) and the fact that if $s \neq \ell$ then $w_{s+1} > w_s$ or $w_{s+1} < w_{r-1}$, we obtain precisely, from the definition, that $\mathrm{ins}(w \setminus \mathrm{sw}(w), \mathrm{sw}(w)) = w$. $\boxed{QED}$

Example 9. Let $a = 1328$ a word and $b = 6754$ a wave. Then $\mathrm{ins}(a, b) = 13675428$ and we have already seen in the previous example that $\mathrm{sw}(13675428) = 6754 = b$.

Thanks to the definition of the special wave of a word w we can evaluate the statistic des in a recursive way.

Proposition 4. *Let $w = w_1 \cdots w_\ell \in W$ be a word whose special wave is* $\mathrm{sw}(w) = w_r w_{r+1} \cdots w_t \cdots w_{s-1} w_s$. *Then we have that*

$$\mathrm{des}(w) = \mathrm{des}(w \setminus \mathrm{sw}(w)) + \mathrm{des}(\mathrm{sw}(w)). \tag{29}$$

Proof. First of all, we note that if $r > 1$ and $s < \ell$ we have that

$$\begin{aligned} \mathrm{des}(w) = \mathrm{des}(w_1 \cdots w_\ell) = & \,\mathrm{des}(w_1 \cdots w_{r-1} w_{s+1} \cdots w_\ell) + \mathrm{des}(w_r \cdots w_s) + \\ & - \chi(w_{r-1} > w_{s+1}) + \chi(w_{r-1} > w_r) + \chi(w_s > w_{s+1}), \end{aligned} \tag{30}$$

where $\chi(P)$ equals 1 if the property P is true, and 0 otherwise. From the definition of r we have that $w_{r-1} < w_r$, and so that $\chi(w_{r-1} > w_r) = 0$.

If $w_{s+1} > w_s$, then $w_{s+1} > w_s > w_{r-1}$ and so that $\chi(w_{r-1} > w_{s+1}) = \chi(w_s > w_{s+1}) = 0$. If $w_{s+1} < w_s$, then by definition of s, we have that $w_{s+1} > w_{r-1}$ and so that $\chi(w_{r-1} > w_{s+1}) = \chi(w_s > w_{s+1}) = 1$. In both cases equation (30) simplifies to equation (29).

If $r = 1$ and $s < \ell$ then

$$\begin{aligned}\operatorname{des}(w) = \operatorname{des}(w_1 \cdots w_\ell) &= \operatorname{des}(w_{s+1} \cdots w_\ell) + \operatorname{des}(w_1 \cdots w_s) + \chi(w_s > w_{s+1}) \\ &= \operatorname{des}(w_{s+1} \cdots w_\ell) + \operatorname{des}(w_1 \cdots w_s) \\ &= \operatorname{des}(w \setminus \operatorname{sw}(w)) + \operatorname{des}(\operatorname{sw}(w)),\end{aligned}$$

since by definition of s in this case $w_s < w_{s+1}$.

If $r > 1$ and $s = \ell$ then

$$\begin{aligned}\operatorname{des}(w) = \operatorname{des}(w_1 \cdots w_\ell) &= \operatorname{des}(w_1 \cdots w_{r-1}) + \operatorname{des}(w_r \cdots w_\ell) + \chi(w_r > w_{r-1}) \\ &= \operatorname{des}(w_1 \cdots w_{r-1}) + \operatorname{des}(w_r \cdots w_\ell) \\ &= \operatorname{des}(w \setminus \operatorname{sw}(w)) + \operatorname{des}(\operatorname{sw}(w)),\end{aligned}$$

since by definition of r it holds that $w_{r-1} < w_r$. from the definition of $\operatorname{sw}(w)$, we have that $w_{r-1} < w_r$ or $r = 1$.

Finally if $r = 1$ and $s = \ell$, then $w = \operatorname{sw}(w)$, $w \setminus \operatorname{sw}(w)$ is the empty word, and then equation (29) is easily true. $\boxed{QED}$

Example 10. Let $\sigma = 13675428 \in \mathcal{S}_8 \subset W$. We have that

$$\begin{aligned}\operatorname{des}(\sigma) &= \operatorname{des}(13675428) = 3 = 1 + 2 \\ &= \operatorname{des}(1328) + \operatorname{des}(6754) = \operatorname{des}(\sigma \setminus \operatorname{sw}(\sigma)) + \operatorname{des}(\operatorname{sw}(\sigma)).\end{aligned}$$

Remark 12. Similarly to what was observed in Remarks 4 and 6, if a wave β has length ℓ, then we have that $1 \le \operatorname{des}(\beta) \le \ell - 1$. Moreover if we have ℓ distinct numbers $b_1 < b_2 < \cdots < b_\ell$, for any $0 \le d \le \ell - 1$, we have exactly one way to rearrange them to create a wave, possibly trivial if $d = 0$, β_d with $\operatorname{des}(\beta_d)$ equal to d, that is, $\beta_d = b_{d+1} b_{d+2} \cdots b_\ell b_d b_{d-1} \cdots b_1$.

Now we can define the desired bijection.

Let $\psi : W \to W$ be the map such that:

- If $w = \varepsilon$ is the empty word then $\psi(\varepsilon) = \varepsilon$.
- Otherwise
$$\psi(w) = \psi(w \setminus \operatorname{sw}(w)) * \delta(\operatorname{sw}(w)) \tag{31}$$

where the $*$ operator is the concatenation of words and $\delta(\mathrm{sw}(w)) = c_{d+1}c_1c_2\cdots c_dc_{d+2}\cdots c_m$ is a rearrangement of $\mathrm{sw}(w)$, where $c_1 < c_2 < \cdots < c_m$ are the elements of $\mathrm{sw}(w)$ and $d = \mathrm{des}(\mathrm{sw}(w))$.

Theorem 5. *The map $\psi : W \to W$ just defined is a well defined bijection of W, such that*

$$\mathrm{des}(w) = \mathrm{lec}(\psi(w)). \tag{32}$$

Moreover the restriction to the symmetric group $\psi_{\mathcal{S}_n} : \mathcal{S}_n \to \mathcal{S}_n$ is a bijection.

Proof. We will prove by induction on the length ℓ of $w \in W$ that $\psi(w)$ is a rearrangement of w, and that it is a well defined map. If $\ell = 0$ then $w = \varepsilon$ is the empty word and by definition $\psi(\varepsilon) = \varepsilon$, that is fine. If w has length $\ell > 0$, then by definition $w \setminus \mathrm{sw}(w)$ is a word of length strictly less than ℓ, and so by the inductive hypothesis $\psi(w \setminus \mathrm{sw}(w))$ is a rearrangement of $w \setminus \mathrm{sw}(w)$. Moreover $\delta(\mathrm{sw}(w))$ is defined as a rearrangement of $\mathrm{sw}(w)$, so this means that $\psi(w)$ is a rearrangement of w, and ψ is a well defined map.

To prove that ψ is a bijection we will define its inverse φ as the map such that:

- If $w = \varepsilon$ is the empty word then $\varphi(\varepsilon) = \varepsilon$.
- Otherwise, if the rightmost hook $\mathrm{h}(w)$ of w is not trivial

$$\varphi(w) = \mathrm{ins}(\varphi(w \setminus \mathrm{h}(w)), \mathrm{wv}(\mathrm{h}(w))), \tag{33}$$

  where $w \setminus \mathrm{h}(w)$ is the prefix of w obtained removing $\mathrm{h}(w)$ and $\mathrm{wv}(\mathrm{h}(w)) = c_{d+1}c_{d+2}\cdots c_mc_dc_{d-1}\cdots c_1$ is a rearrangement of $\mathrm{h}(w)$, where $c_1 < c_2 < \cdots < c_m$ are the elements of $\mathrm{h}(w)$ and $d = \mathrm{lec}(\mathrm{h}(w)) = \mathrm{inv}(\mathrm{h}(w))$. If $\mathrm{h}(w)$ is a trivial hook, then $\mathrm{h}(w) = w$ and in this case $\varphi(w) = w$

First of all let us note that if α is a hook with $\mathrm{lec}(\alpha) = d$, then, thanks to Remark 12, we have that $\mathrm{wv}(\alpha)$ is the unique wave such that its elements are the same of α and such that $\mathrm{des}(\mathrm{wv}(\alpha)) = \mathrm{lec}(\alpha) = d$. Similarly, if β is a wave with $\mathrm{des}(\beta) = d$, then, thanks to Remark 4, we have that $\delta(\beta)$ is the unique hook such that its elements are the same of β and such that $\mathrm{lec}(\delta(\beta)) = \mathrm{des}(\beta) = d$. Putting these two last observations together, we obtain, if α is a hook and β is a wave, that

$$\delta(\mathrm{wv}(\alpha)) = \alpha \tag{34}$$

$$\mathrm{wv}(\delta(\beta)) = \beta. \tag{35}$$

We can notice that from the definition of ψ we have that if w is increasing then $\mathrm{sw}(w) = w$ and $\psi(w) = w$, and so it is clear that ψ and φ are inverse when restricted to the subset of increasing words, because they are both the identity in this case.

Now let us consider the case where $w \in W$ is not increasing.

Since $w \backslash \mathrm{h}(w)$ has length strictly less than w and, by definition, $\mathrm{wv}(\mathrm{h}(w))$ is a wave, using inductive reasoning, entirely analogous to what was done previously for ψ, we obtain that $\varphi(w)$ is a rearrangement of w and that it is a well defined map.

Let us prove that they are inverse maps by induction on the length ℓ of w. If $\ell = 0$ it is clear. If $\ell > 0$ then

$$\begin{aligned}\psi(\varphi(w)) &= \psi(\mathrm{ins}(\varphi(w \setminus \mathrm{h}(w)), \mathrm{wv}(\mathrm{h}(w)))) \\ &= \psi(\varphi(w \setminus \mathrm{h}(w))) * \delta(\mathrm{wv}(\mathrm{h}(w))) \\ &= w \setminus \mathrm{h}(w) * \mathrm{h}(w) = w,\end{aligned} \tag{36}$$

where the second equality is true thanks to Proposition 3 and the last equality is true thanks to inductive hypothesis, since $w \setminus \mathrm{h}(w)$ has length strictly less than ℓ, and thanks to equation (34). Moreover

$$\begin{aligned}\varphi(\psi(w)) &= \mathrm{ins}(\varphi(\psi(w) \setminus \mathrm{h}(\psi(w))), \mathrm{wv}(\mathrm{h}(\psi(w)))) \\ &= \mathrm{ins}(\varphi(\psi(w) \setminus \delta(\mathrm{sw}(w))), \mathrm{wv}(\delta(\mathrm{sw}(w)))) \\ &= \mathrm{ins}(\varphi(\psi(w \setminus \mathrm{sw}(w))), \mathrm{sw}(w)) \\ &= \mathrm{ins}(w \setminus \mathrm{sw}(w)), \mathrm{sw}(w)) = w,\end{aligned} \tag{37}$$

where the second equality is true since the rightmost hook of $\psi(w)$ is by definition $\delta(\mathrm{sw}(\psi(w)))$, the third equality is true thanks to equation (35) and thanks to the definition of $\psi(w)$. The fourth equality holds by inductive hypothesis, since the length of $w \setminus \mathrm{sw}(w)$ is strictly less than ℓ and the last is true thanks to Proposition 3. We have proved that ψ and φ are inverse, and so that they are bijections of W. Moreover we have already proved that they permute the elements of the words, so that their restrictions to the symmetric group $\mathcal{S}_n$ are two bijections.

We can also show the identity of equation (32) by an inductive argument on the length ℓ of $w \in W$. If $\ell = 0$, then $w = \varepsilon$ is the empty word and $\mathrm{lec}(\psi(\varepsilon)) = \mathrm{lec}(\varepsilon) = 0 = \mathrm{des}(\varepsilon$. If w has length $\ell > 0$ we have that

$$\begin{aligned}\mathrm{lec}(\psi(w)) &= \mathrm{lec}(\psi(w \setminus \mathrm{sw}(w))) + \mathrm{lec}(\delta(\mathrm{sw}(w)) \\ &= \mathrm{des}(w \setminus \mathrm{sw}(w)) + \mathrm{des}(\mathrm{sw}(w)) = \mathrm{des}(w),\end{aligned} \tag{38}$$

where the first equality holds since $\delta(\mathrm{sw}(w))$ is the rightmost hook of $\psi(w)$, the second is true by inductive hypothesis and by definition of δ. The last equality is true thanks to Proposition 4. $\boxed{QED}$

Example 11. Let $\sigma = 13675428 \in \mathcal{S}_8 \subset W$, then

$$\begin{aligned}\psi(13675428) &= \psi(1328) * \delta(6754) = \psi(18) * \delta(32) * 6457 \\ &= \psi(\varepsilon) + \delta(18) * 32 * 6457 = \varepsilon * 18 * 32 * 6457 \\ &= 18326457\end{aligned}$$

and

$$\begin{aligned}\varphi(18326457) &= \mathrm{ins}(\varphi(1832), \mathrm{wv}(6457)) = \mathrm{ins}(\mathrm{ins}(\varphi(18), \mathrm{wv}(32)), 6754) \\ &= \mathrm{ins}(\mathrm{ins}(18, 32)), 6754) = \mathrm{ins}(1328, 6754) = 13675428.\end{aligned}$$

Moreover $\mathrm{des}(13675428) = 3 = \mathrm{lec}(18326457) = \mathrm{lec}(\psi(13675428))$.

Remark 13. If $w = \gamma * \alpha_1 * \cdots * \alpha_k$ is the Hook factorization of a word $w \in W$, then we can write

$$w = \mathrm{conc}(\cdots \mathrm{conc}(\mathrm{conc}(\gamma, \alpha_1), \alpha_2) \cdots, \alpha_k),$$

where conc is the binary operation that concatenates words, i.e. $\mathrm{conc}(w', w'') = w' * w''$ for all $w', w'' \in W$.

We have that the map φ is nothing more than the map obtained by replacing each conc with ins, and each hook α_i with its corresponding wave $\mathrm{wv}(\alpha_i)$, namely:

$$\varphi(w) = \mathrm{ins}(\cdots \mathrm{ins}(\mathrm{ins}(\gamma, \mathrm{sw}(\alpha_1)), \mathrm{sw}(\alpha_2)) \cdots, \mathrm{sw}(\alpha_k)).$$

Theorem 5 therefore also implies that every word in W can be obtained, in a unique manner, through an ordered and finite sequence of insertions of waves, starting from an increasing word. The map ψ will then be analogously the one that replaces waves β_i with the corresponding hooks $\delta(\beta_i)$ and insertion operations with concatenation operations.

In particular, the set of permutations $\sigma \in \mathcal{S}_n$ with $\mathrm{lec}(\sigma) = i$ and whose rightmost hook has length at least $n-k$ is in bijection, via φ and ψ, with the set of permutations $\sigma \in \mathcal{S}_n$ with $\mathrm{des}(\sigma) = i$ and whose special wave has length at least $n-k$.

This result improves Corollary 1:

Theorem 6. *Let $0 \leq k \leq n-1$ and $i \geq 0$. The following sets have the same cardinality:*

- *the set of the permutations of $k+1$ different numbers in $[n]$ with i descents,*
- *the set of permutations $\sigma \in \mathcal{S}_n$ with i descents and whose special reverse wave has length at least $n-k$,*

- *the set of permutations* $\sigma \in \mathcal{S}_n$ *with* i *descents and whose special wave has length at least* $n-k$*,*
- *the set of permutations* $\sigma \in \mathcal{S}_n$ *with* $\operatorname{lec}(\sigma) = i$ *and whose rightmost hook has length at least* $n-k$*.*

Remark 14. The cardinality of the previous sets is

$$\sum_{d=n-k}^{n} \binom{n}{d} \sum_{\ell=1}^{d-1} E(n-d, i-\ell), \tag{39}$$

where $E(a,b) = \#\{\, \sigma \in \mathcal{S}_a \mid \operatorname{lec}(\sigma) = b \,\}$ are the Eulerian numbers.

Indeed, from the third characterization, we can count these permutations by first choosing the d elements and the value ℓ of the lec of the rightmost hook.

We can also rephrase this as

$$\sum_{d=0}^{k} \binom{n}{d} \sum_{\ell=1}^{n-d-1} E(d, i-\ell). \tag{40}$$

Acknowledgements. The authors would like to thank Michele D'Adderio for helpful discussions on this issue and for his valuable advice.

References

[1] Timothy Chow: *A new generalization of Eulerian numbers*, URL (version: 2026-01-30): `https://mathoverflow.net/q/489302`.

[2] C. De Concini, C. Procesi: *Wonderful models of subspace arrangements*, Selecta Mathematica, New Series **1**, 459–494 (1995).

[3] D. C. Foata and G.-N. Han: *Fix-Mahonian calculus. III. A quadruple distribution*, Monatsh. Math. **154** (2008), no. 3, 177–197; MR2413301.

[4] G. Gaiffi: *Blowups and cohomology bases for De Concini-Procesi models of subspace arrangements*, Selecta Math. (N.S.) **3** (1997), no. 3, 315–333; MR1481132.

[5] G. Gaiffi, O. Papini, V. Siconolfi: *A basis for the cohomology of compact models of toric arrangements*, Pure Appl. Math. Q. **20** (2024), no. 1, 427–470; MR4734875.

[6] I. M. Gessel: *A coloring problem*, Amer. Math. Monthly **98** (1991), no. 6, 530–533; MR1109577.

[7] J.-L. Lin: *The geometry and combinatorics of some Hessenberg varieties related to the permutohedral variety*, Electron. J. Combin. **31** (2024), no. 3, Paper No. 3.17, 23 pp.; MR4790337.

[8] OEIS Foundation Inc. (2026): Entry A381706 in The On-Line Encyclopedia of Integer Sequences, `https://oeis.org/A381706`.

[9] C. Procesi: *The toric variety associated to Weyl chambers*, Mots: mélanges offerts à M.-P. Schützenberger, Lang. Raison. Calc., 153–161, 1990; MR1252659.