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Some results on automorphisms of finite p-groups

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Abstract. Let G be a group and $\mathrm{Aut}^{\Phi}(G)$ denote the normal subgroup of $\mathrm{Aut}(G)$ consisting of all automorphisms centralizing $G/\Phi(G)$. As a well-known result in a finite p-group G, the group $\mathrm{Aut}^{\Phi}(G)$ is inner if and only if G is extraspecial or elementary abelian. In the present paper, first we give a new proof of this result. We then introduce a new subgroup of $\mathrm{Aut}^{\Phi}(G)$ for a finite p-group G, and give some basic aspects of this subgroup.

Keywords: Automorphism group; central automorphism; inner automorphism; finite p-group

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Introduction

Throughout the paper all groups are assumed to be finite and p a prime number. We denote by $G', Z(G), \Phi(G)$ and $\mathrm{Aut}(G)$, the derived subgroup, the center, the Frattini subgroup and the group of all automorphisms of G, respectively. Similarly to [13], let N be a characteristic subgroup of G and let $\mathrm{Aut}^{N}(G)$ be the subgroup of $\mathrm{Aut}(G)$ consisting of all automorphisms which induces the identity automorphisms on G/N or equivalently, $[x, \alpha] = x^{-1}\alpha(x) \in N$ for all $x \in G$ and $\alpha \in \mathrm{Aut}^{N}(G)$. Now let M be a normal subgroup of G and $C_{\mathrm{Aut}^{N}(G)}(M)$ denote the group of all automorphisms of $\mathrm{Aut}^{N}(G)$ centralizing M. If we choose $N = G'$ or $N = Z(G)$, then $\mathrm{Aut}^{N}(G)$ is precisely the group of all derival automorphisms or central automorphisms of G. Moreover, if $N = \Phi(G)$, then $\mathrm{Aut}^{\Phi}(G)$, where $\Phi = \Phi(G)$ is a normal subgroup of $\mathrm{Aut}(G)$ containing $\mathrm{Inn}(G)$, the group of all inner automorphisms of G.

Müller in [13, Proposition 3.1 and main theorem] proved the following famous results.

Theorem 1. *Let G be a finite non-abelian p-group. Then $C_{\mathrm{Aut}^{\Phi}(G)}(Z(G)) = \mathrm{Inn}(G)$ if and only if $\Phi(G) \leq Z(G)$ and $\Phi(G)$ is cyclic.*

Theorem 2. *If G is a finite p-group which is neither elementary abelian nor extraspecial, then* $\mathrm{Aut}^{\Phi}(G)/\mathrm{Inn}(G)$ *is a nontrivial normal p-subgroup of the* *group of outer automorphisms of G.*

In the first section of this paper we give new proofs of Müller's results. Also by defining a new subgroup of $\mathrm{Aut}^{\Phi}(G)$ for a finite p-group G, we give some basic aspects of this subgroup.
The terms of the lower and the upper central series of a group G are denoted by $\Gamma_i(G)$ and $Z_i(G)$, respectively. For a finite p-group G, $\Omega_1(G) = \langle g \in G \mid g^p = 1\rangle$. Moreover $d(G)$ and $\mathcal{M}(G)$ denote the minimal number of generators and the set of all maximal subgroups of G, respectively. Finally, we use $\mathrm{Hom}(A, B)$ to denote the group of all homomorphisms of A into an abelian group B. Clearly, $\mathrm{Hom}(A, B)$ forms an abelian group under the following operation $(fg)(x) = f(x)g(x)$, for all $f, g \in \mathrm{Hom}(A, B)$ and $x \in G$. Any unexplained notations are standard and can be found in [7].

1 Proofs of the Müller's results

The following two lemmas are well-known results and can be verified easily.

Lemma 1. *Let G be a group and M, N be normal subgroups of G with $N \leq M$ and $C_N(M) \leq Z(G)$. Then*

$$C_{\mathrm{Aut}^N(G)}(M) \cong \mathrm{Hom}(G/M, C_N(M)).$$

Lemma 2. *Let G be a finite p-group, M be a maximal subgroup of G and $g \in G \setminus M$. Let $u \in Z(M)$ such that $(gu)^p = g^p$. Then the map $m \mapsto m$, $g \mapsto gu$ can be extended to an automorphism of G leaving M elementwise fixed.*

In the following proof, we replace substantial parts of Müller's original proofs by new arguments.

Proof of Theorem 1. First we assume that $\Phi(G) \leq Z(G)$ and $\Phi(G)$ is cyclic. Hence by Lemma 1,

$$C_{\mathrm{Aut}^{\Phi}(G)}(Z(G)) \cong \mathrm{Hom}(G/Z(G), \Phi(G)) \cong G/Z(G),$$

and so $C_{\mathrm{Aut}^{\Phi}(G)}(Z(G)) = \mathrm{Inn}(G)$, as required.

Conversely, let $C_{\mathrm{Aut}^{\Phi}(G)}(Z(G)) = \mathrm{Inn}(G)$. We consider two cases:
Case I. $Z(G) \nleq \Phi(G)$.
Assume either $\Phi(G) \nleq Z(G)$ or $\Phi(G)$ is noncyclic. Let M be a maximal subgroup of G such that $Z(G) \nleq M$. So $G = MZ(G)$ and $Z(M) = Z(G) \bigcap M$. Now if $\Phi(M) \leq Z(M)$, then $\Phi(G) \leq Z(G)$ and hence by Lemma 1,

$$\mathrm{Inn}(G) = C_{\mathrm{Aut}^{\Phi}(G)}(Z(G)) \cong \mathrm{Hom}(G/Z(G), \Phi(G)).$$

It follows that $\Phi(G)$ is cyclic, which is impossible.

So we suppose that $\Phi(M) \nleq Z(M)$. In this situation, by using induction we deduce that $C_{\mathrm{Aut}^{\Phi}(M)}(Z(M)) \neq \mathrm{Inn}(M)$. We write $G = M\langle z\rangle$ where $z \in Z(G)\setminus M$. Let $\alpha \in C_{\mathrm{Aut}^{\Phi}(M)}(Z(M))\setminus\mathrm{Inn}(M)$ and extend α to an automorphism $\hat{\alpha} \in C_{\mathrm{Aut}^{\Phi}(G)}(Z(G))$ by setting $\hat{\alpha}(hz^i) = \alpha(h)z^i$, where $h \in M$ and $0 \leq i < p$. We therefore have $\hat{\alpha} \in \mathrm{Inn}(G)$, and so $\alpha \in \mathrm{Inn}(M)$ is a contradiction.

Case II. $Z(G) \leq \Phi(G)$.
Hence $C_G(M) = Z(M)$, for every $M \in \mathcal{M}(G)$. Let H be the subgroup of G such that $H/Z(G) = \Omega_1(Z_2(G)/Z(G))$. Clearly $H \leq Z_2(G)$ and $[H, G] \leq \Omega_1(Z(G))$. First, we prove that $C_G(H) = \Phi(G)$. Let $M \in \mathcal{M}(G)$, $g \in G \setminus M$ and $1 \neq u \in \Omega_1(Z(G))$. Then by Lemma 2, the map α given by $g \mapsto gu$ and $m \mapsto m$, for all $m \in M$, defines a central automorphism of G, which fixes $\Phi(G)$ elementwise. Since $C_{\mathrm{Aut}^{Z}(G)}(\Phi(G)) \leq C_{\mathrm{Aut}^{\Phi}(G)}(Z(G))$, it follows that α is an inner automorphism of G induced by an element t_M in G. Hence $t_M \in C_G(M) = Z(M)$ and $u = [g, t_M] \in Z(G)$. Next, we claim that $t_M \in H$. To see this, we say since $G = M\langle g\rangle$, for any $x \in G$, $x = mg^i$ for some $m \in M$ and $i \in \{0, ..., p-1\}$. As $t_M \in C_G(M)$, we have $[t_M, mg^i] = [t_M, g^i] = [t_M, g]^i \in Z(G)$, which implies that $t_M \in Z_2(G)$. Similarly, $[{t_M}^p, mg^i] = 1$, which shows that ${t_M}^p \in Z(G)$ and so $t_M \in H$. Since $M \leq C_G(t_M)$, if $C_G(t_M) = G$ then $t_M \in Z(G)$ and $gu = \alpha(g) = t_M^{-1}gt_M = g$, whence $u = 1$, which is impossible. Therefore $M = C_G(t_M)$, where $t_M \in H$ and so $C_G(H) \leq M$. Thus $C_G(H) \leq \bigcap_{M\in\mathcal{M}(G)} M = \Phi(G)$. Moreover, $[H, \Phi(G)] = 1$ shows that $\Phi(G) \leq C_G(H)$, from which we find that $C_G(H) = \Phi(G)$, as desired. To continue the proof, we consider two cases:

Case I. $C_G(Z(\Phi(G))) = \Phi(G)$. It follows that $C_G(\Phi(G)) = Z(\Phi(G))$. Now $H \leq C_G(\Phi(G))$ and so $H \leq Z(\Phi(G))$. Next, let X be a maximal abelian normal subgroup of G containing H. Then $X = C_G(X) \leq C_G(H) = \Phi(G)$. It follows that $Z(\Phi(G)) \leq C_G(X) = X$. Also since the following extension $X \rightarrowtail G \twoheadrightarrow G/X$ does not split, it follows that by [9, Corollary 10.1.14], the group $H^2(G/X, X) \neq 0$, whence by [9, Theorem 10.1.12], and a result of [6], that

$$C_{\mathrm{Aut}^{X}(G)}(X)/(C_{\mathrm{Aut}^{X}(G)}(X) \cap \mathrm{Inn}(G)) \cong H^1(G/X, X) \neq 0.$$

Therefore $C_{\mathrm{Aut}^{X}(G)}(X)$ is not contained in $\mathrm{Inn}(G)$, whence $C_{\mathrm{Aut}^{\Phi}(G)}(Z(\Phi(G))) \nleq \mathrm{Inn}(G)$, which is a contradiction, since $C_{\mathrm{Aut}^{\Phi}(G)}(Z(\Phi(G))) \leq C_{\mathrm{Aut}^{\Phi}(G)}(Z(G))$.

Case II. $C_G(Z(\Phi(G))) \neq \Phi(G)$. We show that H is non-abelian; otherwise $H \leq C_G(H) = \Phi(G)$ and so $H \leq Z(\Phi(G))$, because $[H, \Phi(G)] = 1$, which yields the contradiction

$$\Phi(G) \leq C_G(Z(\Phi(G))) \leq C_G(H) = \Phi(G).$$

Next, we show that $Z(G)$ is cyclic. For any $t \in H$, the map $f_t : \bar{H} \rightarrow \Omega_1(Z(G))$ defined by $f_t(\bar{u}) = [u, t]$ is a homomorphism, where $\bar{u}$ and $\overline{H}$ denote the coset $u\Phi(G)$ and the quotient group $H\Phi(G)/\Phi(G)$, respectively. Also the map $f : H \to \mathrm{Hom}(\bar{H}, \Omega_1(Z(G)))$ given by $t \mapsto f_t$ is a homomorphism, with $\mathrm{Ker}(f) = H \cap C_G(H) = H \cap \Phi(G)$. Next, we prove that f is an epimorphism. To see this, let $\varphi \in \mathrm{Hom}(\bar{H}, \Omega_1(Z(G)))$. Also let $\{\bar{x_1}, ..., \bar{x_r}\}$ be a minimal generating set for $\bar{H}$ and if necessary, we extend this to a minimal generating set $\{\bar{x_1}, ..., \bar{x_d}\}$ of $\bar{G}$, where $d = d(G)$. Since $\bar{G}$ is elementary abelian, the homomorphism φ can be extended to the homomorphism $\theta : \bar{G} \to \Omega_1(Z(G))$ defined by

$$\theta(\bar{x_i}) = \begin{cases} \varphi(\bar{x_i}) & 1 \leq i \leq r \\ 1 & r < i \leq d \end{cases}$$

Now, we define $\alpha = \alpha_\theta : G \to G$ by $\alpha(x) = x\theta(\bar{x})$. Since $x^{-1}\alpha(x) \in \Omega_1(Z(G)) \leq \Phi(G)$, for all $x \in G$, we may write G as the product of the image of α and the Frattini subgroup of G and so the image of α must be G itself. Thus α is an automorphism of G. We have $\alpha \in C_{\mathrm{Aut}^\Phi(G)}(Z(G))$. By hypothesis, $\alpha = \gamma_a$ is an inner automorphism of G induced by an element a in G. We observe that $a \in H$. Therefore for each $h \in H$,

$$\varphi(\bar{h}) = \theta(\bar{h}) = h^{-1}\alpha(h) = h^{-1}\gamma_a(h) = [h, a] = f_a(\bar{h}),$$

whence $f(a) = f_a = \varphi$, which shows that f is an epimorphism. It follows that

$$\bar{H} \cong H/(H \cap \Phi(G)) \cong \mathrm{Hom}(\bar{H}, \Omega_1(Z(G))),$$

and hence $\Omega_1(Z(G)) \cong C_p$, which implies that $Z(G)$ is cyclic, as desired.

Finally, we show that $\Phi(G) \leq Z(G)$. Since H is non-abelian, there exist $a, b \in H \setminus Z(G)$ such that $[a, b] \neq 1$. Take $K = \langle a, b\rangle$ and put $L = C_G(K)$. We observe that $[H, G] = H' = \Omega_1(Z(G)) = \langle [a, b]\rangle$. Next, for any $x \in G$, $[a, x] = [a, b]^s$ and $[b, x] = [a, b]^t$ for some integers s, t. Then $[a, b^{-s}a^t x] = 1$ and $[b, b^{-s}a^t x] = 1$. Hence $b^{-s}a^t x \in C_G(K)$ and so G is the central product of K and L. Moreover, $Z(K) = \langle [a, b], a^p, b^p\rangle = \Phi(K)$. We see that $L < G$, $K/Z(K) \cong C_p \times C_p$ where C_p is the cyclic group of order p, $Z(G) = Z(L)$ and $K \cap L = Z(K)$. Furthermore, $\Phi(L) \leq Z(L)$ if and only if $\Phi(G) \leq Z(G)$. Now, we prove that $\Phi(G) \leq Z(G)$. Let $\alpha \in C_{\mathrm{Aut}^\Phi(L)}(Z(L))$. Since $K \cap L \leq Z(G)$, we can extend α to an automorphism $\hat{\alpha} \in C_{\mathrm{Aut}^\Phi(G)}(Z(G))$ with the map $\hat{\alpha} : xy \mapsto x\alpha(y)$, where $x \in K$ and $y \in L$. Then by assumption $\hat{\alpha} \in \mathrm{Inn}(G)$. It follows that α is an inner automorphism of L and we have $C_{\mathrm{Aut}^\Phi(L)}(Z(L)) = \mathrm{Inn}(L)$. Now, we may use induction to deduce that $\Phi(L) \leq Z(L)$, whence $\Phi(G) \leq Z(G)$, as required. The proof of the theorem is now complete.

Let G be a finite p-group. If $\mathrm{Aut}^\Phi(G) = 1$, then G is abelian, since $\mathrm{Inn}(G) \leq \mathrm{Aut}^\Phi(G)$. Hence by [17, Lemma 2.4], $\Phi(G) = 1$, which shows that G is elementary abelian. So for a finite p-group G, $\mathrm{Aut}^\Phi(G) = 1$ if and only if G is elementary abelian. Therefore Theorem 2 is equivalent to the following purpose: for a finite p-group G, $\mathrm{Aut}^{\Phi}(G) = \mathrm{Inn}(G)$ if and only if G is extraspecial.

Proof of Theorem 2. If $\mathrm{Aut}^{\Phi}(G) = \mathrm{Inn}(G)$, then $\Phi(G) \leq Z(G)$ and $\Phi(G)$ is cyclic by Theorem 1. So G is of class 2 and $G' \cong C_p$. Next, assume that $|\Phi(G) : G'| = p^e$. Then $\Phi(G) \cong C_{p^{e+1}}$. Since $\exp(G/G') \leq p^{e+1}$, we have

$$|\mathrm{Aut}^{\Phi}(G)| = |\mathrm{Hom}(G/G', \Phi(G))| = |G/G'| = |G|/p,$$

by [17, Lemma 2.4] and so $G' = Z(G) \cong C_p$. It follows that $Z(G) = \Phi(G)$ and hence G is extra special.

Conversely, as $\exp(G') = \exp(G/Z(G)) = \exp(G/G')$ we have

$$\mathrm{Aut}^{\Phi}(G) = \mathrm{Aut}^{G'}(G) \cong \mathrm{Hom}(G/G', G') \cong G/G' = G/Z(G),$$

since $\mathrm{Aut}^{G'}(G) \cong \mathrm{Hom}(G/G', G')$ by using Lemma 1. Now the result follows from the fact that $\mathrm{Inn}(G) \leq \mathrm{Aut}^{\Phi}(G)$.

2 Some results on the Frattini absolute centre of a group

Hegarty [8], generalized the concept of centre into absolute centre $L(G)$ of a group G as the set

$$L(G) = \{g \in G \mid [g, \alpha] = 1, \forall \alpha \in \mathrm{Aut}(G)\},$$

and the subgroup of G generated by all $[g, \alpha]$ (where $g \in G$ and $\alpha \in \mathrm{Aut}(G)$) is called the autocommutator subgroup of G. In recent years, numerous works have explored the properties of these subgroups and the relations among them. Meng and Guo [11], investigate the relationship between absolute centre $L(G)$ and the Frattini subgroup $\Phi(G)$, for a finite group G, and then they describe the structure of finite groups G satisfying $L(G) \leq \Phi(G)$. Brescia [2], shown that the torsion subgroup of the absolute centre of G lies almost always inside the Frattini subgroup of G and give polynomial bounds for the order of the autocommutator subgroup of G. Following these studies, various authors have investigated the subgroup of automorphisms of G that centralizing $G/L(G)$ under many properties, among which we find [14, 15, 16, 18] and [19]. Following Ellis [4], a group M is said to be a G-group if G is a group acting on M via a homomorphism $\phi : G \to \mathrm{Aut}(M)$ with image containing $\mathrm{Inn}(M)$. Accordingly, the G-centre of M is defined as

$$Z_G(M) = \{m \in M \mid m^g = m, \forall g \in G\},$$

and its G-commutator subgroup is

$$[M,G] = \langle m^{-1}m^g \mid m \in M, g \in G\rangle.$$

Moreover, following [3], let

$$Var_G(M) = \{g \in G \mid m^{-1}m^g \in Z_G(M), \forall m \in M\}.$$

We observe that $Z_M(M)$ is the centre of M, while $Z_{\mathrm{Aut}(M)}(M)$ is the absolute centre of M. Brescia et al. [1], from the G-centre of M, constructed the upper G-central series of M:

$$1 = Z_{G,0}(M) \leq Z_{G,1}(M) \leq \cdots \leq Z_{G,n}(M) \leq \cdots,$$

by the rule $Z_{G,i+1}(M)/Z_{G,i}(M) = Z_G(M/Z_{G,i}(M))$. On the other hand, the lower G-central series of M:

$$M = \gamma_{G,1}(M) \geq \gamma_{G,2}(M) \geq \cdots \geq \gamma_{G,n}(M) \geq \cdots,$$

is defined by $\gamma_{G,i+1}(M) = [\gamma_{G,i}(M), G]$. Clearly, if $G = \mathrm{Aut}(M)$, then $\gamma_{G,2}(M)$ is the autocommutator of M. Next, if we put $A = \mathrm{Aut}^{\Phi}(G)$, then the A-centre of G, is the set $L_f(G)$, consisting of all elements g of G fixed by every automorphism of $\mathrm{Aut}^{\Phi}(G)$, which we call the Frattini absolute centre of G. In fact,

$$L_f(G) = \{g \in G \mid [g,\alpha] = 1, \forall \alpha \in \mathrm{Aut}^{\Phi}(G)\}.$$

It is easy to check that the Frattini absolute centre of a group G is a central characteristic subgroup of G. Moreover, the second term of the lower A-central series of G is the subgroup of G generated by all $[g,\alpha]$, where $g \in G$ and $\alpha \in \mathrm{Aut}^{\Phi}(G)$. Finally,

$$Var_A(G) = \{\alpha \in \mathrm{Aut}^{\Phi}(G) \mid [g,\alpha] \in L_f(G), \forall g \in G\},$$

which we denote it by $\mathrm{Aut}_f(G)$.

Now let G be a finite p-group of class c such that $\Phi(G) \neq 1$, $\exp(\Gamma_c(G)) = p^m$ and $N = \Gamma_c^{p^{m-1}}$, the group generated by all p^{m-1}-th powers of elements of Γ_c. Since by [10, Theorem 2], $C_{\mathrm{Aut}^{\Phi}(G)}(N) = \mathrm{Aut}^{\Phi}(G)$, it follows that $N \leq L_f(G) \cap \Phi(G)$ and so the Frattini absolute centre of G is non-trivial. Next, let $C_{\mathrm{Aut}^{\Phi}(G)}(\mathrm{Aut}_f(G))$ be the centralizer of $\mathrm{Aut}_f(G)$ in $\mathrm{Aut}^{\Phi}(G)$ and

$$K_f(G) = \langle [g,\alpha] \mid g \in G, \alpha \in C_{\mathrm{Aut}^{\Phi}(G)}(\mathrm{Aut}_f(G))\rangle.$$

Since every central automorphism commutes with each inner automorphism of G, it follows that $\mathrm{Inn}(G) \leq C_{\mathrm{Aut}^{\Phi}(G)}(\mathrm{Aut}_f(G))$. Obviously, $K_f(G)$ is a characteristic subgroup in G, which is containing the derived subgroup $G' = \langle [g,\alpha] \mid g \in G, \alpha \in \mathrm{Inn}(G)\rangle$. For the rest of the paper, we assume that G is a finite p-group such that $\Phi(G) \neq 1$.

We begin by stating a number of results that will be used the sequel.

Lemma 3. *Let G be a group. Then $C_{\mathrm{Aut}_f(G)}(K_f(G)) = \mathrm{Aut}_f(G)$.*

Proof. Let $\alpha \in \mathrm{Aut}_f(G)$. Then $x^{-1}\alpha(x) \in L_f(G)$ for all $x \in G$ and so $\alpha(x) = xl_x$, for some $l_x \in L_f(G)$. By selecting an automorphism $\beta \in C_{\mathrm{Aut}^{\Phi}(G)}(\mathrm{Aut}_f(G))$, we observe that

$$\alpha(x^{-1}\beta(x)) = \alpha(x^{-1})\alpha\beta(x) = \alpha(x^{-1})\beta\alpha(x)$$

$$= l_x^{-1}x^{-1}\beta(x)\beta(l_x) = x^{-1}\beta(x)l_x^{-1}l_x = x^{-1}\beta(x),$$

which completes the proof. $\boxed{QED}$

Corollary 1. *Let G be a finite p-group. Then $\mathrm{Aut}_f(G) \leq C_{\mathrm{Aut}^{\Phi}(G)}(\mathrm{Aut}^{K_f}(G))$.*

Proof. Let $\alpha \in \mathrm{Aut}_f(G), \beta \in \mathrm{Aut}^{K_f}(G)$ and $x \in G$. Since $C_{\mathrm{Aut}^{K_f}(G)}(L_f(G)) = \mathrm{Aut}^{K_f}(G)$ we have

$$\beta\alpha(x) = \beta(xl) = \beta(x)l = xkl,$$

for some $l \in L_f(G)$ and $k \in K_f(G)$. On the other hand by applying this fact that every element of $\mathrm{Aut}_f(G)$ fixes the subgroup $K_f(G)$ of G elementwise, we conclude that

$$\alpha\beta(x) = \alpha(xk) = xl\alpha(k) = xlk.$$

Hence $\alpha\beta = \beta\alpha$ and so $\mathrm{Aut}_f(G) \leq C_{\mathrm{Aut}^{\Phi}(G)}(\mathrm{Aut}^{K_f}(G))$, as required. $\boxed{QED}$

Notation. Let G be a finite p-group such that $G/G' = \langle \bar{x_1} \rangle \times \cdots \times \langle \bar{x_r} \rangle$ with $\bar{x_i} = x_iG'$, $\Phi(G) \leq Z(G)$ and $u \in \Phi(G)$ such that $o(u) \mid o(\bar{x_s})$, for some $1 \leq s \leq r$. We define the map f by $\bar{x_s} \mapsto u$, which is trivial on the complement of $\langle \bar{x_s} \rangle$ and denote it by $f_{x_s,u}$. We observe that $f_{x_s,u} \in \mathrm{Hom}(G/G', \Phi(G))$.

Lemma 4. *Let G be a finite p-group.*

(i) $L_f(G) \leq \Phi(G)$.

(ii) If $\Phi(G) \leq Z(G)$, then $L_f(G) = G'G^{p^n}$, where $\exp(\Phi(G)) = p^n$.

Proof. (i) Assume that there exists a maximal subgroup M of G such that $L_f(G) \not\leq M$. We choose an element z in $\Omega_1(L_f(G) \cap \Phi(G))$ and $h \in L_f(G) \setminus M$. Hence $G = M\langle h \rangle$. Then it is an easy task to check that the map α defined by $\alpha(mh^i) = m(hz)^i$, for every $m \in M$ and $0 \leq i < p$, belongs to $\mathrm{Aut}^{\Phi}(G)$. Since $h \in L_f(G)$, it follows from the definition of $L_f(G)$ that $h = \alpha(h) = hz$. Hence $z = 1$, which is a contradiction.

(ii) Let $x \in G$ and $\alpha \in \mathrm{Aut}^{\Phi}(G)$. Since $x^{-1}\alpha(x) \in \Phi(G)$, $\alpha(x) = xz$ for some $z \in \Phi(G)$. Hence $\alpha(x^{p^n}) = (xz)^{p^n} = x^{p^n}$ because $\Phi(G) \leq Z(G)$ and $\exp(\Phi(G)) = p^n$, which shows that $\mathrm{Aut}^{\Phi}(G)$ acts trivially on $G'G^{p^n}$. So $G'G^{p^n} \leq$

$L_f(G)$. We claim that $L_f(G) = G'G^{p^n}$; otherwise $G'G^{p^n} < L_f(G)$ and so there exists x in $L_f(G)$ which is not in $G'G^{p^n}$. Let $\bar{G} = G/G' = \langle \bar{x_1} \rangle \times \cdots \times \langle \bar{x_r} \rangle$, where $x_1, ..., x_r \in G$ and $\bar{x} = xG' = x_1^{p^{t_1}} ... x_r^{p^{t_r}} G'$ for suitable $t_i \geq 0$. Thus $x_j^{p^{t_j}} G' \notin G'G^{p^n}$ for some j and therefore $t_j < n$. Now we choose an element $u \in \Phi(G)$ such that $o(u) = \min\{o(\bar{x_j}), p^n\}$. Then by taking the homomorphism $f_{x_j,u}$, the map $\alpha(g) = g f_{x_j,u}(\bar{g})$, for all $g \in G$ is an endomorphism of G. Since $g^{-1}\alpha(g) \in \Phi(G)$, it follows that $G = \mathrm{Im}(\alpha)\Phi(G)$, whence $G = \mathrm{Im}(\alpha)$ and α is an epimorphism, which implies that $\alpha \in \mathrm{Aut}^{\Phi}(G)$. Next, since $\alpha(x) = x f_{x_j,u}(\bar{x})$, $x \in L_f(G)$ and

$$f_{x_j,u}(\bar{x}) = f_{x_j,u}(\bar{x_j}^{p^{t_j}}) = u^{p^{t_j}},$$

it follows that $u^{p^{t_j}} = 1$. Now if $o(u) = p^n$, then $n \leq t_j < n$, a contradiction. Therefore $o(u) = o(\bar{x_j})$ and $o(\bar{x_j}) \mid p^{t_j}$, whence $x_j^{p^{t_j}} G' = G' \leq G^{p^n} G'$, which is impossible. $\boxed{QED}$

Lemma 5. *Let G be a group. Then*

(i) $\mathrm{Aut}_f(G) \cong \mathrm{Hom}(G/K_f(G)L_f(G), L_f(G)) \cong \mathrm{Hom}(G, L_f(G))$.

(ii) $C_{\mathrm{Aut}_f(G)}(Z(G)) \cong \mathrm{Hom}(G/K_f(G)Z(G), L_f(G))$.

Proof. (i) Let $\theta \in \mathrm{Aut}_f(G)$. Then $f_\theta : gK_f(G)L_f(G) \mapsto g^{-1}\theta(g)$, defines a homomorphism from $G/K_f(G)L_f(G)$ to $L_f(G)$ and the map φ sending θ to f_θ defines a monomorphism from $\mathrm{Aut}_f(G)$ to $\mathrm{Hom}(G/K_f(G)L_f(G), L_f(G))$. On the other hand, by considering $f \in \mathrm{Hom}(G/K_f(G)L_f(G), L_f(G))$, the map $\theta = \theta_f$ defined by $\theta(x) = x f(xK_f(G)L_f(G))$, for all $x \in G$, belongs to $\mathrm{Aut}_f(G)$ and $\varphi(\theta) = \varphi(\theta_f) = f$. Hence φ is onto and so

$$\mathrm{Aut}_f(G) \cong \mathrm{Hom}(G/K_f(G)L_f(G), L_f(G)).$$

Next, define the following map

$$\varphi : \mathrm{Aut}_f(G) \to \mathrm{Hom}(G, L_f(G))$$

$$\alpha \mapsto f_\alpha,$$

where $f_\alpha : G \to L_f(G)$ given by $f_\alpha(x) = x^{-1}\alpha(x)$, for all $x \in G$. Since every automorphism of $\mathrm{Aut}^{\Phi}(G)$ acts trivially on the Frattini absolute centre of G, it follows that f_α is a well-defined homomorphism from G to $L_f(G)$. Obviously, φ is well-defined, an injective map and it is a homomorphism, since for all $\beta, \gamma \in \mathrm{Aut}_f(G)$ and $x \in G$, we have

$$f_{\beta\gamma}(x) = x^{-1}\beta\gamma(x) = x^{-1}\beta(xx^{-1}\gamma(x)) = x^{-1}\beta(x)x^{-1}\gamma(x),$$

because $x^{-1}\gamma(x) \in L_f(G)$. Thus $f_{\beta\gamma}(x) = f_\beta(x)f_\gamma(x)$ and so φ is a homomorphism. Moreover the homomorphism φ is also surjective, for this let $f \in \mathrm{Hom}(G, L_f(G))$. Then the map $\alpha = \alpha_f$ defined by $\alpha(x) = xf(x)$ for all $x \in G$ is an endomorphism of G. Since by Lemma 4, $x^{-1}\alpha(x) \in L_f(G) \leq \Phi(G)$ for all $x \in G$, we may write G as the product of the image of α and the Frattini subgroup of G and so the image of α must be G itself. Thus $\alpha_f \in \mathrm{Aut}_f(G)$ and then it is evident that $f_{\alpha_f} = f$, which shows that φ is an isomorphism and the result holds.

(ii) It is sufficient to observe that for each $\theta \in C_{\mathrm{Aut}_f(G)}(Z(G))$, the map

$$f_\theta : G/K_f(G)Z(G) \to L_f(G)$$

given by $f_\theta(xK_f(G)Z(G)) = x^{-1}\theta(x)$, for all $x \in G$, is a homomorphism and $\theta \mapsto f_\theta$ is an isomorphism from $C_{\mathrm{Aut}_f(G)}(Z(G))$ to $\mathrm{Hom}(G/K_f(G)Z(G), L_f(G))$. $\boxed{QED}$

We note that if $A \leq Z(G)$ and $X/K_f(G)L_f(G)$ is a direct factor of the group $G/K_f(G)L_f(G)$, then any element $f \in \mathrm{Hom}(X/K_f(G)L_f(G), A)$ induces an element $\bar{f} \in \mathrm{Hom}(G/K_f(G)L_f(G), A)$ which is trivial on the complement of the group $X/K_f(G)L_f(G)$ in $G/K_f(G)L_f(G)$. To simplify the notation, in the following lemma, we shall identify f with the corresponding homomorphism from G into X which is induced by $\bar{f}$.

Lemma 6. *Let G be a finite p-group. Then $\mathrm{Aut}_f(G)$ is elementary abelian if and only if $\exp(L_f(G)) = p$ or $\exp(G/K_f(G)L_f(G)) = p$.*

Proof. First we assume that $\mathrm{Aut}_f(G)$ is an elementary abelian group such that $\exp(L_f(G)) > p$ and $\exp(G/K_f(G)L_f(G)) > p$. Furthermore, suppose that $\langle l \rangle$ and $\langle \bar{g} \rangle$, $\bar{g} = gK_f(G)L_f(G)$ are cyclic direct factors of $L_f(G)$ and $G/K_f(G)L_f(G)$ with maximum possible orders, respectively. Define the subgroup

$$R = \{\alpha_f : f \in \mathrm{Hom}(\langle \bar{g} \rangle, \langle l \rangle)\}$$

of $\mathrm{Aut}_f(G)$, where $\alpha_f(x) = xf(x)$, for all $x \in G$. We observe that R is a cyclic subgroup of $\mathrm{Aut}_f(G)$ of order at least p^2, which is impossible.

Conversely, if $\exp(L_f(G)) = p$ or $\exp(G/K_f(G)L_f(G)) = p$, then by Lemma 5, the result follows at once. $\boxed{QED}$

Lemma 7. *Let G be a finite non-abelian p-group. Then $C_{\mathrm{Aut}_f(G)}(Z(G)) = \mathrm{Inn}(G)$ if and only if $G/L_f(G)$ is abelian, $K_f(G) \leq Z(G)$ and $L_f(G)$ is cyclic.*

Proof. First assume that $G/L_f(G)$ is abelian, $K_f(G) \leq Z(G)$ and $L_f(G)$ is cyclic. Since $G' \leq L_f(G)$, it follows that G is of class 2 and by [12, Lemma 0.4],

$\exp(G') = \exp(G/Z(G))$. Hence $\exp(G/Z(G))$ divides $\exp(L_f(G))$ and thus

$$\begin{aligned} C_{\mathrm{Aut}_f(G)}(Z(G)) &\cong \mathrm{Hom}(G/K_f(G)Z(G), L_f(G)) \\ &= \mathrm{Hom}(G/Z(G), L_f(G)) \cong G/Z(G), \end{aligned}$$

which implies that $C_{\mathrm{Aut}_f(G)}(Z(G)) = \mathrm{Inn}(G)$.

Conversely, suppose that $C_{\mathrm{Aut}_f(G)}(Z(G)) = \mathrm{Inn}(G)$. Hence $G' \leq L_f(G)$ and $G/L_f(G)$ is abelian. By Lemma 3, we have

$$\mathrm{Inn}(G) = C_{\mathrm{Aut}_f(G)}(Z(G)) \leq \mathrm{Aut}_f(G) = C_{\mathrm{Aut}_f(G)}(K_f(G)),$$

which shows that $K_f(G) \leq Z(G)$. Now from the fact that

$$\exp(G/Z(G)) = \exp(G') \leq \exp(L_f(G))$$

and

$$\begin{aligned} C_{\mathrm{Aut}_f(G)}(Z(G)) &\cong \mathrm{Hom}(G/K_f(G)Z(G), L_f(G)) \\ &= \mathrm{Hom}(G/Z(G), L_f(G)) \cong G/Z(G), \end{aligned}$$

we deduced that $L_f(G)$ is cyclic, as required. $\boxed{QED}$

The following conclusion from [20, Corollary 2.2] plays an important role in the rest of the paper.

Lemma 8. *Let A, B and C be finite abelian p-groups, $\exp(C) = p^t$ and $A \leq B$. Then $\mathrm{Hom}(A, C) \cong \mathrm{Hom}(B, C)$ if and only if $A \cong H \times A_1, B \cong H \times B_1$ where all invariants of A_1, B_1 are at least t, $d(A_1) = d(B_1)$ and $\exp(H) < p^t$.*

Theorem 3. *Let G be a finite non-abelian p-group. Then the following statements are equivalent:*

(i) $\mathrm{Aut}_f(G) = C_{\mathrm{Aut}_f(G)}(Z(G))$;

(ii) $Z(G) \leq K_f(G)G^{p^n}$, *where* $\exp(L_f(G)) = p^n$;

(iii) One of the following statements holds:

(a) $Z(G) \leq K_f(G)$ *or*

(b) $G/K_f(G)Z(G) \cong A \times X_1, G/G' \cong A \times Y_1$, *where all invariants of* X_1, Y_1 *are at least* n, *with* $\exp(L_f(G)) = p^n$, $d(X_1) = d(Y_1)$ *and* $\exp(A) < p^n$.

Proof. We show that (i)$\leftrightarrow$ (ii) and first assume that $\mathrm{Aut}_f(G) = C_{\mathrm{Aut}_f(G)}(Z(G))$. If $Z(G) \nleq K_f(G)G^{p^n}$ where $\exp(L_f(G)) = p^n$, then $K_f(G)G^{p^n} < K_f(G)Z(G)G^{p^n}$. Next, as

$$G' \leq K_f(G) \leq K_f(G)G^{p^n} < K_f(G)Z(G)G^{p^n},$$

it follows that

$$\mathrm{Hom}(G/K_f(G)Z(G)G^{p^n}, L_f(G)) \hookrightarrow \mathrm{Hom}(G/K_f(G)G^{p^n}, L_f(G))$$

$$\hookrightarrow \mathrm{Hom}(G/K_f(G), L_f(G)) \hookrightarrow \mathrm{Hom}(G/G', L_f(G)) \cong \mathrm{Aut}_f(G),$$

by Lemma 5. On the other hand

$$\mathrm{Aut}_f(G) = C_{\mathrm{Aut}_f(G)}(Z(G)G^{p^n}) \cong \mathrm{Hom}(G/K_f(G)Z(G)G^{p^n}, L_f(G)).$$

Thus we deduce that

$$\mathrm{Hom}(G/K_f(G)Z(G)G^{p^n}, L_f(G)) \cong \mathrm{Hom}(G/K_f(G)G^{p^n}, L_f(G)).$$

Next, by Lemma 8, we may assume that

$$G/K_f(G)Z(G)G^{p^n} \cong A \times X_1, G/K_f(G)G^{p^n} \cong A \times Y_1,$$

where all invariants of X_1, Y_1 are at least n, $d(X_1) = d(Y_1)$ and $\exp(A) < p^n$. Since $\exp(G/K_f(G)Z(G)G^{p^n})$ and $\exp(G/K_f(G)G^{p^n})$ are at most n, it follows that all invariants of X_1 and Y_1 are exactly equal to n and so $X_1 = Y_1$, because $d(X_1) = d(Y_1)$ and $G/K_f(G)Z(G)G^{p^n}$ is isomorphic to a subgroup of $G/K_f(G)G^{p^n}$. Hence $G/K_f(G)Z(G)G^{p^n} \cong G/K_f(G)G^{p^n}$ and thus $K_f(G)Z(G)G^{p^n} = K_f(G)G^{p^n}$, which is impossible.

Conversely, if $Z(G) \leq K_f(G)G^{p^n}$, then since every automorphisms of $\mathrm{Aut}_f(G)$ fixing $K_f(G)$ and G^{p^n} elementwise, we deduce that $\mathrm{Aut}_f(G) = C_{\mathrm{Aut}_f(G)}(Z(G))$.

(i)$\leftrightarrow$ (iii) First assume that $\mathrm{Aut}_f(G) = C_{\mathrm{Aut}_f(G)}(Z(G))$ and $Z(G) \nleq K_f(G)$. Then

$$\mathrm{Hom}(G/G', L_f(G)) \cong \mathrm{Hom}(G/K_f(G)Z(G), L_f(G)),$$

and by Lemma 8, $G/K_f(G)Z(G) \cong A \times X_1$, $G/G' \cong A \times Y_1$, where all invariants of X_1, Y_1 are at least n, with $\exp(L_f(G)) = p^n$, $d(X_1) = d(Y_1)$ and $\exp(A) < p^n$.

Conversely, if (a) holds and $Z(G) \leq K_f(G)$, then $\mathrm{Aut}_f(G) = C_{\mathrm{Aut}_f(G)}(Z(G))$, by Lemma 3. In the second case, by applying Lemma 8,

$$\mathrm{Hom}(G/K_f(G)Z(G), L_f(G)) \cong \mathrm{Hom}(G/G', L_f(G)).$$

It follows that $|C_{\mathrm{Aut}_f(G)}(Z(G))| = |\mathrm{Aut}_f(G)|$ by Lemma 5, and so the result holds. $\boxed{QED}$

Theorem 4. *Let G be a finite p-group such that $G/L_f(G)$ is abelian. Then the following statements are equivalent:*

(i) $\mathrm{Aut}_f(G) = C_{\mathrm{Aut}_f(G)}(Z(G))$*;*

(ii) $K_f(G) \leq Z(G)$, $G/Z(G) \cong A \times X_1, G/G' \cong A \times Y_1$, *where all invariants of* X_1 *are exactly equal to* n *with* $\exp(L_f(G)) = p^n$, $d(X_1) = d(Y_1)$ *and* $\exp(A) < p^n$*;*

(iii) $Z(G) = K_f(G)G^{p^n}$, *where* $\exp(L_f(G)) = p^n$.

Proof. (i) $\Rightarrow$ (ii) Let G be a p-group such that $\mathrm{Aut}_f(G) = C_{\mathrm{Aut}_f(G)}(Z(G))$. This implies that $\mathrm{Hom}(G/G', L_f(G)) \cong \mathrm{Hom}(G/K_f(G)Z(G), L_f(G))$. Since $G/L_f(G)$ is abelian, it follows that $\mathrm{Inn}(G) \leq \mathrm{Aut}_f(G) = C_{\mathrm{Aut}_f(G)}(K_f(G))$, whence $K_f(G) \leq Z(G)$ and so $\mathrm{Hom}(G/G', L_f(G)) \cong \mathrm{Hom}(G/Z(G), L_f(G))$. Next, by Lemma 8, $G/Z(G) \cong A \times X_1$ and $G/G' \cong A \times Y_1$, where all invariants of X_1, Y_1 are at least n, with $\exp(L_f(G)) = p^n$, $\exp(A) < p^n$ and $d(X_1) = d(Y_1)$. Since

$$\exp(G/Z(G)) = \exp(G') \leq \exp(L_f(G)) = p^n,$$

it follows that all invariants of X_1 are exactly equal to n, and so the result holds.

(ii) $\Rightarrow$ (iii) Since by the assumption $G/Z(G) \cong X_1 \times A$, where all invariants of X_1 are exactly equal to n and $\exp(A) < p^n$, it follows that $\exp(G/Z(G)) = p^n$ and so $K_f(G)G^{p^n} \leq Z(G)$. This implies that $G/Z(G)$ is a quotient subgroup of $G/K_f(G)G^{p^n}$, which shows that $\exp(G/K_f(G)G^{p^n}) = p^n$. To continue the proof, we may assume that $G/K_f(G)G^{p^n}$ and A are of types $(b_1, b_2, ..., b_t)$ and $(a_{r+1}, a_{r+2}, ..., a_t)$ respectively, where

$$d(G/Z(G)) = d(G/K_f(G)G^{p^n}) = d(G/G') = t$$

and $d(X_1) = r$. Now since

$$G/Z(G) \hookrightarrow G/K_f(G)G^{p^n} \hookrightarrow G/G',$$

also $\exp(G/K_f(G)G^{p^n}) = p^n$, $G/Z(G) \cong X_1 \times A$ where $d(X_1) = r$ and all invariants of X_1 are exactly equal to n, it follows that $b_i = n$ for $1 \leq i \leq r$. Moreover, $a_i \leq b_i \leq a_i$ for $r+1 \leq i \leq t$, because $d(Y_1) = d(X_1) = r$ and $G/G' \cong Y_1 \times A$. It follows that $a_i = b_i$ for $r+1 \leq i \leq t$. Hence $G/Z(G) = G/K_f(G)G^{p^n}$ and consequently $Z(G) = K_f(G)G^{p^n}$, as desired.

(iii) $\Rightarrow$ (i) Since $Z(G) = K_f(G)G^{p^n}$, Theorem 3 implies the result. $\boxed{QED}$

Theorem 5. *Let G be a finite group such that $Z(G/L_f(G)) = R/L_f(G)$. Then* *(i)* $Z(\mathrm{Inn}(G)) \leq \mathrm{Aut}_f(G)$ *if and only if* $R = Z_2(G)$.

(ii) $\mathrm{Aut}_f(G) = Z(\mathrm{Inn}(G))$ *if and only if* $\mathrm{Hom}(G/K_f(G)L_f(G), L_f(G)) \cong R/Z(G)$ *and* $R = Z_2(G)$.

(iii) $\mathrm{Aut}_f(G) \leq Z(\mathrm{Inn}(G))$ *if and only if* $\mathrm{Hom}(G/K_f(G)L_f(G), L_f(G)) \cong R/Z(G)$.

(iv) *If* $\mathrm{Aut}_f(G) \leq Z(\mathrm{Inn}(G))$ *then* $Z(G) \leq K_f(G)G^{p^n}$, *where* $\exp(L_f(G)) = p^n$.

Proof. (i) First suppose that $Z(\mathrm{Inn}(G)) \leq \mathrm{Aut}_f(G)$ and let $r \in R$. Then for all $g \in G$, $[g, r] \in L_f(G) \leq Z(G)$, since $Z(G/L_f(G)) = R/L_f(G)$, whence $r \in Z_2(G)$. Next, suppose that $r \in Z_2(G)$ and γ_r is the inner automorphism of G induced by r. Then for any $g \in G$, $[g, \gamma_r] = g^{-1}r^{-1}gr = [g, r] \in [G, Z_2(G)] \leq Z(G)$. Thus $\gamma_r \in \mathrm{Aut}^Z(G)$ and hence $\gamma_r \in Z(\mathrm{Inn}(G)) \leq \mathrm{Aut}_f(G)$. Therefore $[g, r] = [g, \gamma_r] \in L_f(G)$, for all $g \in G$, whence $rL_f(G) \in Z(G/L_f(G))$ and $r \in R$, which shows that $R = Z_2(G)$ and the result holds.

For the converse, suppose that $R = Z_2(G)$ and let $\gamma_r \in Z(\mathrm{Inn}(G))$, the inner automorphism of G induced by r. Hence $\gamma_r \in \mathrm{Aut}^Z(G)$ and so $[g, r] = [g, \gamma_r] \in Z(G)$, for all $g \in G$. It follows that $r \in Z_2(G) = R$, whence $rL_f(G) \in R/L_f(G) = Z(G/L_f(G))$. Therefore $[g, \gamma_r] = [g, r] \in L_f(G)$, for any $g \in G$ and so $\gamma_r \in \mathrm{Aut}_f(G)$ and the result holds again.

(ii) Suppose that $\mathrm{Aut}_f(G) = Z(\mathrm{Inn}(G))$. Then by (i) $R = Z_2(G)$ and by Lemma 5,

$$\mathrm{Hom}(G/K_f(G)L_f(G), L_f(G)) \cong R/Z(G).$$

Conversely, suppose that $\mathrm{Hom}(G/K_f(G)L_f(G), L_f(G)) \cong R/Z(G)$ and $R = Z_2(G)$. Then by Lemma 5,

$$\mathrm{Aut}_f(G) \cong R/Z(G) = Z_2(G)/Z(G) = Z(\mathrm{Inn}(G)).$$

Hence $|\mathrm{Aut}_f(G)| = |Z(\mathrm{Inn}(G))|$, which together with (i), $\mathrm{Aut}_f(G) = Z(\mathrm{Inn}(G))$.

(iii) First we observe that $Z(G) \leq R$, because if $z \in Z(G)$ then $zL_f(G) \in Z(G/L_f(G)) = R/L_f(G)$ and thus $z \in R$. Next we show that $R/Z(G) \cong \mathrm{Aut}_f(G) \cap Z(\mathrm{Inn}(G))$. For this, suppose that $r \in R$. Then $\gamma_g\gamma_{r^{-1}} = \gamma_{r^{-1}}\gamma_g$ for any $g \in G$, since

$$\gamma_g\gamma_{r^{-1}}(x) = g^{-1}rxr^{-1}g = [g, r^{-1}]rg^{-1}xr^{-1}g = rg^{-1}xr^{-1}g[g, r^{-1}] = \gamma_{r^{-1}}\gamma_g(x),$$

for all $x \in G$, because $[g, r^{-1}] = [r^{-1}, g]^{-1} \in L_f(G) \leq Z(G)$, whence $\gamma_{r^{-1}} \in \mathrm{Aut}_f(G) \cap Z(\mathrm{Inn}(G))$. Now by defining the map

$$\psi : R \to \mathrm{Aut}_f(G) \cap Z(\mathrm{Inn}(G))$$

given by $\psi(r) = \gamma_{r^{-1}}$, it is easy to see that ψ is an epimorphism with $\mathrm{Ker}(\psi) = R \cap Z(G) = Z(G)$, and so $R/Z(G) \cong \mathrm{Aut}_f(G) \cap Z(\mathrm{Inn}(G))$.
To continue the proof, suppose that $\mathrm{Aut}_f(G) \leq Z(\mathrm{Inn}(G))$. Then

$$\mathrm{Hom}(G/K_f(G)L_f(G), L_f(G)) \cong \mathrm{Aut}_f(G) = \mathrm{Aut}_f(G) \cap Z(\mathrm{Inn}(G)) \cong R/Z(G).$$

Conversely, assume that $\mathrm{Hom}(G/K_f(G)L_f(G), L_f(G)) \cong R/Z(G)$. Then

$$|\mathrm{Aut}_f(G)| = |\mathrm{Hom}(G/K_f(G)L_f(G), L_f(G))| = |R/Z(G)|.$$

Since $|\mathrm{Aut}_f(G) \cap Z(\mathrm{Inn}(G))| = |R/Z(G)|$, it follows that $\mathrm{Aut}_f(G) \cap Z(\mathrm{Inn}(G)) = \mathrm{Aut}_f(G)$, which shows that $\mathrm{Aut}_f(G) \leq Z(\mathrm{Inn}(G))$, and the result holds.
(iv) This follows immediately from Theorem 3. $\boxed{QED}$

Theorem 6. *Let G be a finite non-abelian p-group. Then the following statements are equivalent:*

(i) $\mathrm{Aut}_f(G) = \mathrm{Inn}(G)$*;*

(ii) $K_f(G) \leq Z(G)$*,* $G/L_f(G)$ *is abelian,* $L_f(G)$ *is cyclic and* $Z(G) = K_f(G)G^{p^n}$*, where* $\exp(L_f(G)) = p^n$*;*

(iii) $G/L_f(G)$ *is abelian and* $\mathrm{Hom}(G/K_f(G)L_f(G), L_f(G)) \cong G/Z(G)$*.*

Proof. (i)$\leftrightarrow$ (ii) First suppose that $\mathrm{Aut}_f(G) = \mathrm{Inn}(G)$. This implies that $G' \leq L_f(G)$ and so $G/L_f(G)$ is abelian. Moreover $C_{\mathrm{Aut}_f(G)}(Z(G)) = \mathrm{Inn}(G)$ and so by Lemma 7, $K_f(G) \leq Z(G)$ and $L_f(G)$ is cyclic. Since $\exp(G/Z(G))$ divides $\exp(L_f(G)) = p^n$, we have $G' \leq K_f(G)G^{p^n} \leq Z(G)$. It follows that

$$\mathrm{Inn}(G) \cong \mathrm{Hom}(G/Z(G), L_f(G))$$

$$\hookrightarrow \mathrm{Hom}(G/K_f(G)G^{p^n}, L_f(G)) \hookrightarrow \mathrm{Hom}(G/G', L_f(G)) \cong \mathrm{Aut}_f(G) = \mathrm{Inn}(G).$$

Thus

$$\mathrm{Hom}(G/Z(G), L_f(G)) \cong \mathrm{Hom}(G/K_f(G)G^{p^n}, L_f(G)).$$

Next, by Lemma 8 we may assume that

$$G/Z(G) \cong A \times X_1, G/K_f(G)G^{p^n} \cong A \times Y_1,$$

where all invariants of X_1, Y_1 are at least n, $d(X_1) = d(Y_1)$ and $\exp(A) < p^n$. Now since $\exp(G/K_f(G)G^{p^n})$ and $\exp(G/Z(G))$ are at most n, it follows that all invariants of X_1, Y_1 are exactly equal to n, and so $X_1 = Y_1$, because $d(X_1) = d(Y_1)$ and $G/Z(G)$ is isomorphic to a quotient subgroup of $G/K_f(G)G^{p^n}$. Hence $G/Z(G) \cong G/K_f(G)G^{p^n}$ and thus $Z(G) = K_f(G)G^{p^n}$.

Conversely, assume that $G/L_f(G)$ is abelian, $K_f(G) \leq Z(G)$, $L_f(G)$ is cyclic and $Z(G) = K_f(G)G^{p^n}$, where $\exp(L_f(G)) = p^n$. By using Lemma 7, $C_{\mathrm{Aut}_f(G)}(Z(G)) = \mathrm{Inn}(G)$. Since $Z(G) = K_f(G)G^{p^n}$, we observe that $\mathrm{Aut}_f(G) \leq C_{\mathrm{Aut}_f(G)}(Z(G)) = \mathrm{Inn}(G)$, which shows that $\mathrm{Aut}_f(G) = \mathrm{Inn}(G)$.

(i)$\leftrightarrow$ (iii) First suppose that $\mathrm{Aut}_f(G) = \mathrm{Inn}(G)$. From $\mathrm{Inn}(G) \leq \mathrm{Aut}_f(G)$, we deduce that $G' \leq L_f(G)$ and $G/L_f(G)$ is abelian. Moreover

$$\mathrm{Aut}_f(G) \cong \mathrm{Hom}(G/K_f(G)L_f(G), L_f(G)) \cong G/Z(G),$$

by Lemma 5 and the result holds.

Conversely, since $G/L_f(G)$ is abelian, it follows that $\mathrm{Inn}(G) \leq \mathrm{Aut}_f(G)$ and $R = G$, where $Z(G/L_f(G)) = R/L_f(G)$. Furthermore

$$\mathrm{Hom}(G/K_f(G)L_f(G), L_f(G)) \cong G/Z(G) = R/Z(G)$$

and hence by Theorem 5(iii), $\mathrm{Aut}_f(G) \leq Z(\mathrm{Inn}(G)) = \mathrm{Inn}(G)$. Therefore $\mathrm{Aut}_f(G) = \mathrm{Inn}(G)$, as desired. $\boxed{QED}$

We conclude the paper by giving an example of a group which supports Theorem 3.

Example 1. Consider $G = \langle f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8\rangle$ be a group with relations:

$$f_5^2, f_8^2, f_7^{-1}f_6^2, f_6^{-1}f_4^2, f_7^{-2}f_8, f_1^2f_4^{-1}, f_2^2f_5, f_3^2f_8, f_1f_3f_2f_1^{-1}f_2^{-1}, [f_3, f_1] = f_8,$$
$$[f_2, f_3], [f_2, f_4].$$

This is the group number 455 in the GAP library of groups of order 2^8 and is of nilpotency class 3, because $Z(G) = \langle f_4, f_6, f_7, f_3f_5, f_8\rangle \cong C_2 \times C_{16}$, $Z_2(G) = \langle f_4, f_6, f_7, f_3f_5, f_8, f_5\rangle$ and $Z_3(G) = G$. In this group $L_f(G) = \langle f_8\rangle \cong C_2$, $G' = \langle f_3, f_8\rangle = \langle f_3\rangle \cong C_4$, and $\Phi(G) = \langle f_3, f_4, f_5, f_6, f_7, f_8\rangle \cong C_2 \times C_2 \times C_{16}$, which shows that $Z(G) \leq \Phi(G)$. Also $\mathrm{Aut}_f(G) \cong \mathrm{Hom}(G/G', L_f(G)) \cong C_2 \times C_2$.

By using GAP [5], we see that $C_{\mathrm{Aut}^{\Phi}(G)}(\mathrm{Aut}_f(G)) = \mathrm{Aut}^{\Phi}(G)$. Also by von Dyck's theorem, the mapping

$f_1 \mapsto f_1f_4, f_2 \mapsto f_2f_3, f_3 \mapsto f_3f_8, f_4 \mapsto f_4f_6, f_5 \mapsto f_5f_8, f_6 \mapsto f_6f_7, f_7 \mapsto f_7f_8, f_8 \mapsto f_8$, and

$f_1 \mapsto f_1f_5, f_2 \mapsto f_2, f_3 \mapsto f_3, f_4 \mapsto f_4f_8$ and $f_i \mapsto f_i$ for $5 \leq i \leq 8$,

are automorphisms in $\mathrm{Aut}^{\Phi}(G)$.

Hence $\Phi(G) = \langle f_3, f_4, f_5, f_6, f_7, f_8\rangle \leq K_f(G)$ and so $K_f(G) = \Phi(G)$, which shows that $C_{\mathrm{Aut}_f(G)}(\Phi(G)) = \mathrm{Aut}_f(G)$ by Lemma 3. Finally as $Z(G) \leq \Phi(G) = K_f(G)$, by Theorem 3, $\mathrm{Aut}_f(G) = C_{\mathrm{Aut}_f(G)}(Z(G))$.

It is easy to see that the following four automorphisms belong to $\mathrm{Aut}_f(G)$:

$\alpha_1(f_1) = f_1f_8, \alpha_1(f_2) = f_2f_8$ and $\alpha_1(f_i) = f_i$ for $3 \leq i \leq 8$,

$\alpha_2(f_1) = f_1, \alpha_2(f_2) = f_2 f_8$ and $\alpha_2(f_i) = f_i$ for $3 \leq i \leq 8$,
$\alpha_3(f_1) = f_1 f_8$ and $\alpha_3(f_i) = f_i$ for $2 \leq i \leq 8$,
$\alpha_4(f_i) = f_i$ for $1 \leq i \leq 8$.

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