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Estimation in power Lindley distributions using balanced joint progressively Type-II censored data

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The balanced joint progressive Type-II (BJPT-II) censoring plan has gained considerable attention in reliability theory and lifetime experiments over the last decade. It is particularly useful when the objective of an experiment is to compare the reliability characteristics of units drawn from different independent production lines (or populations) under identical environmental conditions.

In this article, we study estimation in two power Lindley distributions (PLDs) based on BJPT-II censored samples. The maximum likelihood estimators of the unknown parameters are obtained together with their associated asymptotic confidence intervals. Further, Bayes estimators of the model parameters are derived under informative priors and the linear exponential (LINEX) loss function. Within the Bayesian framework, highest posterior density (HPD) credible intervals for the unknown parameters are also constructed. For the Bayesian computations, Markov chain Monte Carlo (MCMC) techniques are implemented. To illustrate the proposed inferential procedures, a detailed simulation study and analysis of two real datasets are presented. Several optimality criteria are also investigated to determine an optimal BJPT-II censoring plan.

Keywords: Balanced joint progressive Type-II censoring plan; maximum likelihood estimation; Bayes estimation; power Lindley distribution; LINEX loss; MCMC methods.

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1 Introduction

In life-testing experiments and reliability studies, researchers aim to assess the survival times of products (units) placed under test. However, due to experimental costs, limited manpower, time restrictions, and other unavoidable constraints, it is often necessary to terminate the experiment before observing the failure times of all units. Such unobserved units are referred to as *censored units*, and the resulting observations constitute *censored data*. Various censoring plans have been extensively discussed in the statistical literature to optimize cost, time, labor, and related factors.

During the last two decades, *progressive censoring* has received significant attention in lifetime experiments. A progressive censoring scheme allows the experimenter to remove predetermined numbers of surviving units at intermediate failure times, in addition to the terminal point. This scheme was introduced by Cohen (1963). For comprehensive discussions of progressive censoring and its wide-ranging applications, the readers may refer to Balakrishnan and Cramer (2014) and Balakrishnan and Aggarwala (2000).

Most conventional censoring schemes (such as Type-I, Type-II, progressive, random, and hybrid censoring) concern the one-sample problem. However, when the objective is to compare the lifetimes of two or more competing products manufactured under the same facility, *joint censoring plans* become essential. The joint Type-II (JT-II) censoring plan was first introduced by Balakrishnan and Rasouli (2008) for two exponential populations. Building on this, Rasouli and Balakrishnan (2010) and Shafay et al. (2014) extended the work to jointly progressive Type-II (JPT-II) and JT-II censoring schemes for two exponential populations, respectively. The expected number of failures under JPT-II censoring was investigated by Parsi and Bairamov (2009). Estimation procedures for JPT-II censored two Weibull populations were developed by Mondal and Kundu (2019b). Reliability estimation for JT-II censored stress–strength models with Weibull component strengths was examined by Cetinkaya (2021). Joint Type-II censoring for two and k Lindley populations was studied in Krishna and Goel (2022) and Goel and Krishna (2023), respectively.

More recently, Mondal and Kundu (2019a) introduced a new JPT-II censoring plan for two exponential populations and derived inferences for the associated model parameters. Subsequently, Mondal and Kundu (2020) extended this work to the Weibull distribution under the assumption that both populations have equal sample sizes, naming the scheme the *balanced joint progressive Type-II* (BJPT-II) censoring plan. A Bayesian optimal life-testing plan under the BJPT-II censoring scheme was later proposed by Mondal et al. (2020). Compared with the general joint progressive censoring plan, the BJPT-II scheme possesses several attractive analytical properties. In particular, the total number of removals at each failure time is the same across the two populations, which simplifies the likelihood structure and improves inferential tractability.

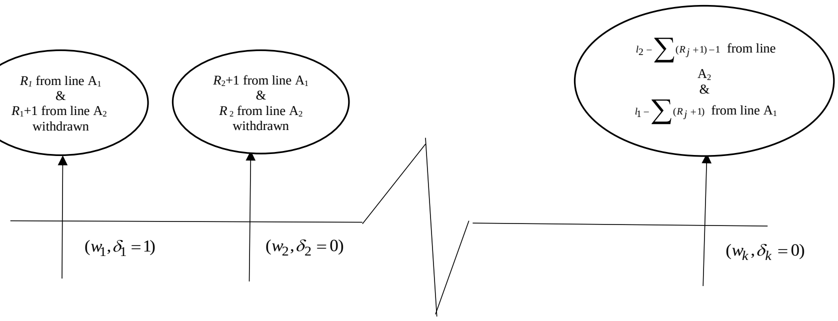
In this article, we focus on estimation problems for two BJPT-II censored populations when the underlying lifetime distributions follow distinct-scale but common-shape *power Lindley distributions* (PLDs). The PLD accommodates a wide range of ageing behaviors, including increasing, decreasing, and bathtub-shaped hazard rate functions, making it a flexible and widely used model in reliability analysis. The distribution was introduced by Ghitany et al. (2013), who studied its fundamental properties and demonstrated its applicability using real data. Subsequent research on the PLD has been extensive: reliability estimation for two and multi component stress–strength models was discussed by Ghitany et al. (2015) and Pak et al. (2018); estimation under progressive censoring was explored by Valiollahi et al. (2018); Bayesian analysis under various loss functions was carried out by Pak et al. (2019); inference on $P(X > Y)$ under progressive Type-II censoring was proposed by Joukar et al. (2020); step-stress partially accelerated testing for progressively censored PLD was developed by Çetinkaya (2021); multicomponent stress–strength reliability estimation under progressive censoring was investigated by Kumari et al. (2024); Saini et al. (2024); discussed estimation of multicomponent stress–strength reliability from PLD under progressive first-failure censored samples; and Shukla et al. (2025) studied Statistical inference and prediction in unified hybrid censoring.

In the present work, the following modelling assumptions are explicitly stated and are used consistently throughout: (i) within each population, lifetimes are independent and identically distributed (i.i.d.) according to a PLD; (ii) the two populations originate from independent production lines but are tested under identical experimental conditions; (iii) both populations share a common shape parameter, representing a common underlying failure mechanism, but have distinct scale parameters to allow different reliability levels; and (iv) the joint removal process strictly adheres to a pre-specified BJPT-II censoring plan. These assumptions play a central role in constructing the likelihood, developing the Bayesian model, and interpreting the real-data analysis.

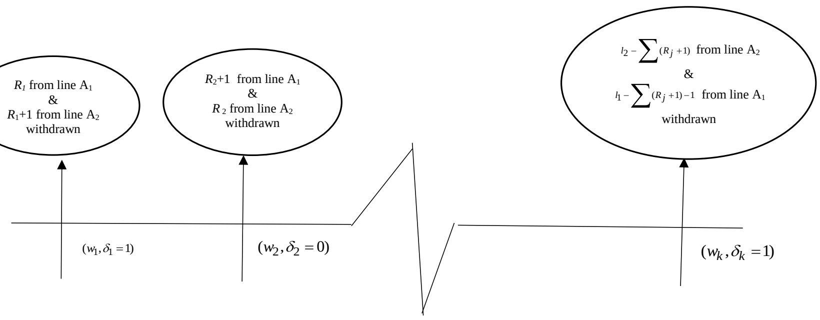
The remainder of the paper is organized as follows. A detailed description of the BJPT-II censoring plan and the associated assumptions is provided in Section 2. Maximum likelihood estimation (MLE) of the unknown parameters, along with their asymptotic confidence intervals, is presented in Section 3. Bayesian estimation under the linear exponential (LINEX) loss function with independent gamma priors is discussed in Section 4. A simulation study illustrating the comparative performance of the estimators is reported in Section 5. A real-data application is analysed in Section 6. Several optimality criteria for determining an optimal BJPT-II scheme are explored in Section 7. Finally, concluding remarks are presented in Section 8.

2 Model description

The BJPT-II censoring plan may be described as follows: Draw two independent random samples of sizes l_1 and l_2 from production lines A_1 and A_2, respectively. Let $k < \min(l_1, l_2)$ be the pre-specified number of failures at which the experiment will be



(a) The kth failure from population A_2



(b) The kth failure from population A_1

Figure 1: Schematic diagram of the BJPT-II censoring plan.

terminated. The vector of integers $\mathbf{R} = (R_1, \ldots, R_{k-1})$ is chosen in advance such that

$$\sum_{i=1}^{k-1} (R_i + 1) < \min(l_1, l_2),$$

ensuring that enough units remain in both populations until the kth failure.

Suppose the first failure occurs at time W_1 from population A_1. At this time, R_1 surviving units are randomly removed from the remaining $l_1 - 1$ units of A_1, and $R_1 + 1$ surviving units are withdrawn from the l_2 units of A_2. If the second failure occurs at time W_2 from population A_2, then R_2 units are removed from the remaining $(l_2 - R_1 - 2)$ units of A_2, while $R_2 + 1$ units are removed from the remaining $(l_1 - R_1 - 1)$ units of A_1, and so on.

This mechanism continues until the kth failure time W_k is observed (originating from either A_1 or A_2), at which point all surviving units from both samples are removed. Let the indicator sequence $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_k)$ be defined as

$$\delta_i = \begin{cases} 1, & W_i \text{ is a failure from population } A_1, \\ 0, & \text{otherwise.} \end{cases}$$

The operational implementation of the BJPT-II plan may be summarized as follows:

1. Fix the sample sizes l_1 and l_2, the terminal failure number k, and the removal vector $\mathbf{R}$ before initiating the test.
2. At each observed failure time W_i, identify the originating population and record the indicator δ_i.
3. Remove R_i surviving units from the population where the failure occurred and remove $R_i + 1$ surviving units from the other population.
4. Continue this process until the kth failure occurs, after which all remaining units are withdrawn.

Figure 1 illustrates the BJPT-II mechanism, showing two possible terminal configurations depending on whether the kth failure arises from A_1 or A_2. At each intermediate failure time, the total number of removals from the two lines remains equal, highlighting the "balanced" feature of the plan.

Let the lifetimes of the l_1 units from production line A_1, $(X_1, X_2, \ldots, X_{l_1})$, be i.i.d. according to the power Lindley distribution (PLD) with scale parameter β_1 and common shape parameter α, denoted by PLD(β_1, α). The corresponding probability density function (pdf) and cumulative distribution function (cdf) are

$$\begin{cases} f_1(x) = \dfrac{\alpha\beta_1^2}{1+\beta_1}\, e^{-\beta_1 x^\alpha}\,(1+x^\alpha)\, x^{\alpha-1}, & x > 0,\ \alpha > 0,\ \beta_1 > 0, \\ F_1(x) = 1 - \dfrac{1+\beta_1+\beta_1 x^\alpha}{1+\beta_1}\, e^{-\beta_1 x^\alpha}, & x > 0,\ \alpha > 0,\ \beta_1 > 0. \end{cases} \tag{1}$$

Similarly, let the lifetimes of the l_2 units from production line A_2, $(Y_1, Y_2, \ldots, Y_{l_2})$, be i.i.d. from PLD$(\beta_2, \alpha)$, with pdf and cdf given by

$$\begin{cases} f_2(y) = \dfrac{\alpha\beta_2^2}{1+\beta_2}\, e^{-\beta_2 y^\alpha}\,(1+y^\alpha)\, y^{\alpha-1}, & y > 0,\ \alpha > 0,\ \beta_2 > 0, \\ F_2(y) = 1 - \dfrac{1+\beta_2+\beta_2 y^\alpha}{1+\beta_2}\, e^{-\beta_2 y^\alpha}, & y > 0,\ \alpha > 0,\ \beta_2 > 0. \end{cases} \tag{2}$$

For the BJPT-II censoring plan, the likelihood function of the observed data $\mathbf{w} =$

$(w_1, \ldots, w_k)$ and indicators $\boldsymbol{\delta}$ is

$$L(\mathbf{w}, \boldsymbol{\delta}, \mathbf{R}; \beta_1, \beta_2, \alpha) = C_1 \prod_{i=1}^{k-1} \Bigg[\left\{ f_1(w_i)\,(1 - F_1(w_i))^{R_i}\,(1 - F_2(w_i))^{R_i+1} \right\}^{\delta_i} \times \left\{ f_2(w_i)\,(1 - F_1(w_i))^{R_i+1}\,(1 - F_2(w_i))^{R_i} \right\}^{1-\delta_i} \Bigg] \times \left[f_1(w_k)\,(1 - F_1(w_k))^{l_1 - \sum_{j=1}^{k-1}(R_j+1) - 1}\,(1 - F_2(w_k))^{l_2 - \sum_{j=1}^{k-1}(R_j+1)} \right]^{\delta_k} \times \left[f_2(w_k)\,(1 - F_2(w_k))^{l_2 - \sum_{j=1}^{k-1}(R_j+1) - 1}\,(1 - F_1(w_k))^{l_1 - \sum_{j=1}^{k-1}(R_j+1)} \right]^{1-\delta_k}, \tag{3}$$

where $C_1 = \prod_{i=1}^{k} \Big[(l_1 - \sum_{j=1}^{i-1}(R_j+1))\,\delta_i + (l_2 - \sum_{j=1}^{i-1}(R_j+1))\,(1-\delta_i) \Big]$, is a normalizing constant.

Although the expression given in equation (3) appears lengthy, it follows directly from the BJPT-II rules: at each failure time, the likelihood contribution consists of the density of the failed unit and the survival probabilities of all withdrawn units. The indicator δ_i determines whether the factors involving (f_1, F_1) or (f_2, F_2) enter the expression at the ith stage.

3 Maximum likelihood estimation

In this section, we consider the maximum likelihood (ML) estimation of the model parameters and derive the corresponding asymptotic confidence intervals. Using the density and distribution expressions in Equations (1) and (2), the likelihood function of the unknown parameters $(\beta_1, \beta_2, \alpha)$, for given l_1, l_2, k, and $(R_1, R_2, \ldots, R_{k-1})$, based on the observed data $(\mathbf{w}, \boldsymbol{\delta}, \mathbf{R})$, can be written as

$$L(\mathbf{w}, \boldsymbol{\delta}, \mathbf{R}; \beta_1, \beta_2, \alpha) = C_1\, \alpha^k\, \frac{\beta_1^{2k_1}}{(1+\beta_1)^{l_1}} \frac{\beta_2^{2k_2}}{(1+\beta_2)^{l_2}} \prod_{i=1}^{k}(1 + w_i^{\alpha}) \prod_{i=1}^{k} w_i^{\alpha-1} \exp(-\beta_1 \xi_1) \exp(-\beta_2 \xi_2) \times \prod_{i=1}^{k-1} \left(1 + \beta_1 + \beta_1 w_i^{\alpha}\right)^{R_i+1-\delta_i} \prod_{i=1}^{k-1} \left(1 + \beta_2 + \beta_2 w_i^{\alpha}\right)^{R_i+\delta_i} \times \left(1 + \beta_1 + \beta_1 w_k^{\alpha}\right)^{l_1 - \sum_{j=1}^{k-1}(R_j+1) - \delta_k} \left(1 + \beta_2 + \beta_2 w_k^{\alpha}\right)^{l_2 - \sum_{j=1}^{k-1}(R_j+1) - 1 + \delta_k}, \tag{4}$$

where C_1 is normalizing constant, and $\xi_1 = \sum_{i=1}^{k-1} w_i^{\alpha}(1 + R_i) + w_k^{\alpha}\Big(l_1 - \sum_{j=1}^{k-1}(R_j+1)\Big)$, $\xi_2 = \sum_{i=1}^{k-1} w_i^{\alpha}(1 + R_i) + w_k^{\alpha}\Big(l_2 - \sum_{j=1}^{k-1}(R_j+1)\Big)$, $k_1 = \sum_{i=1}^{k} \delta_i$, and $k_2 = \sum_{i=1}^{k}(1 - \delta_i)$.

Taking natural logarithms of equation (4), the log-likelihood function is obtained as

$$
\begin{aligned}
\ln L = {} & \ln(C_1) + k\ln\alpha + 2k_1\ln\beta_1 - l_1\ln(1+\beta_1) + 2k_2\ln\beta_2 - l_2\ln(1+\beta_2) + \sum_{i=1}^{k}(\alpha-1)\ln w_i \\
& + \sum_{i=1}^{k}\ln(1+w_i^{\alpha}) - \beta_1\xi_1 - \beta_2\xi_2 + \sum_{i=1}^{k-1}(R_i+1-\delta_i)\ln\big(1+\beta_1+\beta_1 w_i^{\alpha}\big) \\
& + \Big(l_1 - \sum_{j=1}^{k-1}(R_j+1) - \delta_k\Big)\ln\big(1+\beta_1+\beta_1 w_k^{\alpha}\big) + \sum_{i=1}^{k-1}(R_i+\delta_i)\ln\big(1+\beta_2+\beta_2 w_i^{\alpha}\big) \\
& + \Big(l_2 - \sum_{j=1}^{k-1}(R_j+1) - 1 + \delta_k\Big)\ln\big(1+\beta_2+\beta_2 w_k^{\alpha}\big).
\end{aligned}
\tag{5}
$$

Differentiating equation (5) with respect to β_1, β_2, and α yields the nonlinear likelihood equations. The score function with respect to β_1 is

$$
\frac{\partial \ln L}{\partial \beta_1} = \frac{2k_1}{\beta_1} - \frac{l_1}{1+\beta_1} - \xi_1 + \sum_{i=1}^{k-1}\frac{(R_i+1-\delta_i)(1+w_i^{\alpha})}{1+\beta_1+\beta_1 w_i^{\alpha}} + \frac{\big(l_1 - \sum_{j=1}^{k-1}(R_j+1) - \delta_k\big)(1+w_k^{\alpha})}{1+\beta_1+\beta_1 w_k^{\alpha}} = 0.
\tag{6}
$$

Similarly, the score function with respect to β_2 is

$$
\frac{\partial \ln L}{\partial \beta_2} = \frac{2k_2}{\beta_2} - \frac{l_2}{1+\beta_2} - \xi_2 + \sum_{i=1}^{k-1}\frac{(R_i+\delta_i)(1+w_i^{\alpha})}{1+\beta_2+\beta_2 w_i^{\alpha}} + \frac{\big(l_2 - \sum_{j=1}^{k-1}(R_j+1) - 1 + \delta_k\big)(1+w_k^{\alpha})}{1+\beta_2+\beta_2 w_k^{\alpha}} = 0.
\tag{7}
$$

The score function with respect to α is given by

$$
\begin{aligned}
\frac{\partial \ln L}{\partial \alpha} = {} & \frac{k}{\alpha} + \sum_{i=1}^{k}\ln w_i + \sum_{i=1}^{k}\frac{w_i^{\alpha}\ln w_i}{1+w_i^{\alpha}} \\
& + \sum_{i=1}^{k-1}\frac{(R_i+1-\delta_i)\,\beta_1 w_i^{\alpha}\ln w_i}{1+\beta_1+\beta_1 w_i^{\alpha}} + \frac{\big(l_1 - \sum_{j=1}^{k-1}(R_j+1) - \delta_k\big)\,\beta_1 w_k^{\alpha}\ln w_k}{1+\beta_1+\beta_1 w_k^{\alpha}} \\
& + \sum_{i=1}^{k-1}\frac{(R_i+\delta_i)\,\beta_2 w_i^{\alpha}\ln w_i}{1+\beta_2+\beta_2 w_i^{\alpha}} + \frac{\big(l_2 - \sum_{j=1}^{k-1}(R_j+1) - 1 + \delta_k\big)\,\beta_2 w_k^{\alpha}\ln w_k}{1+\beta_2+\beta_2 w_k^{\alpha}} \\
& - \beta_1\left[\sum_{i=1}^{k-1} w_i^{\alpha}\ln w_i\,(1+R_i) + w_k^{\alpha}\ln w_k\Big(l_1 - \sum_{j=1}^{k-1}(R_j+1)\Big)\right] \\
& - \beta_2\left[\sum_{i=1}^{k-1} w_i^{\alpha}\ln w_i\,(1+R_i) + w_k^{\alpha}\ln w_k\Big(l_2 - \sum_{j=1}^{k-1}(R_j+1)\Big)\right] = 0.
\end{aligned}
\tag{8}
$$

The nonlinear equations (6), (7) and (8) do not admit closed-form solutions and must be solved numerically to obtain the ML estimators $\widehat{\beta}_1$, $\widehat{\beta}_2$, and $\widehat{\alpha}$. The `maxLik` package in `R` can be used for numerical optimization. In our computations, we first fit marginal

PLDs to each production line separately to obtain stable initial values and then refine the estimates under the full BJPT-II log likelihood given in equation (5). This strategy improves convergence behavior without changing the final maximizer of of the log-likelihood function given in the equation in (5).

3.1 Asymptotic confidence intervals

In this subsection, we develop a methodology to construct asymptotic confidence intervals (CIs) for the model parameters β_1, β_2, and α. Starting from the log-likelihood function in equation (5), the observed Fisher information matrix, evaluated at the maximum likelihood estimates $(\hat{\beta}_1, \hat{\beta}_2, \hat{\alpha})$, is given by

$$I(\hat{\beta}_1, \hat{\beta}_2, \hat{\alpha}) = -\begin{bmatrix} \frac{\partial^2 \ln L}{\partial \beta_1^2} & \frac{\partial^2 \ln L}{\partial \beta_1 \partial \beta_2} & \frac{\partial^2 \ln L}{\partial \beta_1 \partial \alpha} \\ \frac{\partial^2 \ln L}{\partial \beta_2 \partial \beta_1} & \frac{\partial^2 \ln L}{\partial \beta_2^2} & \frac{\partial^2 \ln L}{\partial \beta_2 \partial \alpha} \\ \frac{\partial^2 \ln L}{\partial \alpha \partial \beta_1} & \frac{\partial^2 \ln L}{\partial \alpha \partial \beta_2} & \frac{\partial^2 \ln L}{\partial \alpha^2} \end{bmatrix}_{\beta_1=\hat{\beta}_1,\ \beta_2=\hat{\beta}_2,\ \alpha=\hat{\alpha}}.$$

After some algebra, the second-order partial derivatives of $\ln L$ are obtained as follows. For β_1,

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial \beta_1^2} &= -\frac{2k_1}{\beta_1^2} + \frac{l_1}{(1+\beta_1)^2} - \sum_{i=1}^{k-1} \frac{(R_i + 1 - \delta_i)}{(1+\beta_1+\beta_1 w_i^\alpha)^2} (1+w_i^\alpha)^2 \\ &\quad - \frac{l_1 - \sum_{j=1}^{k-1}(R_j+1) - \delta_k}{(1+\beta_1+\beta_1 w_k^\alpha)^2} (1+w_k^\alpha)^2. \end{aligned} \tag{9}$$

The mixed derivative with respect to β_1 and β_2 is

$$\frac{\partial^2 \ln L}{\partial \beta_1 \partial \beta_2} = \frac{\partial^2 \ln L}{\partial \beta_2 \partial \beta_1} = 0. \tag{10}$$

For the mixed derivative with respect to β_1 and α, we have

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial \beta_1 \partial \alpha} &= \sum_{i=1}^{k-1} \frac{(R_i + 1 - \delta_i)}{(1+\beta_1+\beta_1 w_i^\alpha)^2} \Big[(1+w_i^\alpha)\, \beta_1 w_i^\alpha \ln(w_i) - (1+\beta_1+\beta_1 w_i^\alpha)\, w_i^\alpha \ln(w_i) \Big] \\ &\quad + \frac{l_1 - \sum_{j=1}^{k-1}(R_j+1) - \delta_k}{(1+\beta_1+\beta_1 w_k^\alpha)^2} \Big[(1+w_k^\alpha)\, \beta_1 w_k^\alpha \ln(w_k) - (1+\beta_1+\beta_1 w_k^\alpha)\, w_k^\alpha \ln(w_k) \Big]. \end{aligned} \tag{11}$$

Similarly, for β_2,

$$\begin{aligned}\frac{\partial^2 \ln L}{\partial \beta_2^2} = -\frac{2k_2}{\beta_2^2} + \frac{l_2}{(1+\beta_2)^2} - \sum_{i=1}^{k-1} \frac{(R_i+\delta_i)}{(1+\beta_2+\beta_2 w_i^\alpha)^2}(1+w_i^\alpha)^2 \\ - \frac{l_2 - \sum_{j=1}^{k-1}(R_j+1) - 1 + \delta_k}{(1+\beta_2+\beta_2 w_k^\alpha)^2}(1+w_k^\alpha)^2.\end{aligned} \tag{12}$$

The mixed derivative with respect to β_2 and α is

$$\begin{aligned}\frac{\partial^2 \ln L}{\partial \beta_2 \partial \alpha} &= \sum_{i=1}^{k-1} \frac{(R_i+\delta_i)}{(1+\beta_2+\beta_2 w_i^\alpha)^2}\Big[(1+w_i^\alpha)\,\beta_2 w_i^\alpha \ln(w_i) - (1+\beta_2+\beta_2 w_i^\alpha)\, w_i^\alpha \ln(w_i)\Big] \\ &+ \frac{l_2 - \sum_{j=1}^{k-1}(R_j+1) - 1 + \delta_k}{(1+\beta_2+\beta_2 w_k^\alpha)^2}\Big[(1+w_k^\alpha)\,\beta_2 w_k^\alpha \ln(w_k) - (1+\beta_2+\beta_2 w_k^\alpha)\, w_k^\alpha \ln(w_k)\Big].\end{aligned} \tag{13}$$

Finally, the second derivative with respect to α is

$$\begin{aligned}\frac{\partial^2 \ln L}{\partial \alpha^2} = &-\frac{k}{\alpha^2} - \sum_{i=1}^{k} \frac{\big(w_i^\alpha \ln(w_i)\big)^2 - (1+w_i^\alpha)\, w_i^\alpha \big(\ln(w_i)\big)^2}{(1+w_i^\alpha)^2} \\ &- \sum_{i=1}^{k-1} \frac{(R_i+1-\delta_i)}{(1+\beta_1+\beta_1 w_i^\alpha)^2}\Big[(\beta_1 w_i^\alpha \ln(w_i))^2 - \beta_1(1+\beta_1+\beta_1 w_i^\alpha)\, w_i^\alpha \big(\ln(w_i)\big)^2\Big] \\ &- \sum_{i=1}^{k-1} \frac{(R_i+\delta_i)}{(1+\beta_2+\beta_2 w_i^\alpha)^2}\Big[(\beta_2 w_i^\alpha \ln(w_i))^2 - \beta_2(1+\beta_2+\beta_2 w_i^\alpha)\, w_i^\alpha \big(\ln(w_i)\big)^2\Big] \\ &- \frac{l_1 - \sum_{j=1}^{k-1}(R_j+1) - \delta_k}{(1+\beta_1+\beta_1 w_k^\alpha)^2}\Big[(\beta_1 w_k^\alpha \ln(w_k))^2 - \beta_1(1+\beta_1+\beta_1 w_k^\alpha)\, w_k^\alpha \big(\ln(w_k)\big)^2\Big] \\ &- \frac{l_2 - \sum_{j=1}^{k-1}(R_j+1) - 1 + \delta_k}{(1+\beta_2+\beta_2 w_k^\alpha)^2}\Big[(\beta_2 w_k^\alpha \ln(w_k))^2 - \beta_2(1+\beta_2+\beta_2 w_k^\alpha)\, w_k^\alpha \big(\ln(w_k)\big)^2\Big] \\ &- \beta_1\left(\sum_{i=1}^{k-1} w_i^\alpha \big(\ln(w_i)\big)^2 (R_i+1) + w_k^\alpha \big(\ln(w_k)\big)^2 \Big[l_1 - \sum_{j=1}^{k-1}(R_j+1)\Big]\right) \\ &- \beta_2\left(\sum_{i=1}^{k-1} w_i^\alpha \big(\ln(w_i)\big)^2 (R_i+1) + w_k^\alpha \big(\ln(w_k)\big)^2 \Big[l_2 - \sum_{j=1}^{k-1}(R_j+1)\Big]\right).\end{aligned} \tag{14}$$

By symmetry of mixed partial derivatives, $\partial^2 \ln L/(\partial\alpha\,\partial\beta_1) = \partial^2 \ln L/(\partial\beta_1\partial\alpha)$ and $\partial^2 \ln L/(\partial\alpha\,\partial\beta_2) = \partial^2 \ln L/(\partial\beta_2\partial\alpha)$, so it is sufficient to use the expressions in equations (11) and (13).

For $0 < \zeta < 1$, the $100(1-\zeta)\%$ asymptotic CIs for β_1, β_2, and α are given by

$$\left(\hat\beta_1 \pm z_{\zeta/2}\sqrt{\widehat{V}(\hat\beta_1)}\right), \qquad \left(\hat\beta_2 \pm z_{\zeta/2}\sqrt{\widehat{V}(\hat\beta_2)}\right), \qquad \left(\hat\alpha \pm z_{\zeta/2}\sqrt{\widehat{V}(\hat\alpha)}\right),$$

where $z_{\zeta/2}$ denotes the $100(1-\zeta/2)^{\text{th}}$ percentile of the standard normal distribution, and $\widehat{V}(\hat{\beta}_1)$, $\widehat{V}(\hat{\beta}_2)$, and $\widehat{V}(\hat{\alpha})$ are the estimated variances of $\hat{\beta}_1$, $\hat{\beta}_2$, and $\hat{\alpha}$, respectively, i.e. the diagonal elements of the inverse observed Fisher information matrix $I(\hat{\beta}_1, \hat{\beta}_2, \hat{\alpha})^{-1}$.

4 Bayesian estimation

In this section, we derive the Bayesian estimators of the unknown model parameters β_1, β_2, and α. Let the prior distributions of β_1, β_2, and α be independent gamma distributions with hyperparameters (a_1, b_1), (a_2, b_2), and (a_3, b_3), respectively. Then the joint prior density of $(\beta_1, \beta_2, \alpha)$ is

$$g(\beta_1, \beta_2, \alpha) \propto \beta_1^{b_1-1} e^{-a_1\beta_1}\, \beta_2^{b_2-1} e^{-a_2\beta_2}\, \alpha^{b_3-1} e^{-a_3\alpha}, \quad \beta_1, \beta_2, \alpha > 0,\ a_i, b_i > 0. \tag{15}$$

The choice of independent gamma priors is motivated by their flexibility on $(0, \infty)$ and their interpretability in terms of prior means and variances. In non-informative or weakly informative settings, the hyperparameters are selected so that the prior variances are large, ensuring that the BJPT-II likelihood dominates the posterior. For informative analyses, the hyperparameters are chosen to reflect plausible reliability levels suggested by engineering judgment or historical experiments. The prior on the common shape parameter α is specified independently of (β_1, β_2), which is consistent with the modelling assumption that the underlying failure mechanism is common across the two production lines. Combining the likelihood in equation (4) with the prior given in the equation (15), the joint posterior density of $(\beta_1, \beta_2, \alpha)$ given the data $(\mathbf{w}, \boldsymbol{\delta}, \mathbf{R})$ is, up to a normalizing constant

$$\begin{aligned}
\pi(\beta_1, \beta_2, \alpha \mid \mathbf{w}, \boldsymbol{\delta}, \mathbf{R}) \propto\ & \frac{\beta_1^{2k_1+b_1-1}}{(1+\beta_1)^{l_1}} \exp\big\{-\beta_1(\xi_1 + a_1)\big\}\, \frac{\beta_2^{2k_2+b_2-1}}{(1+\beta_2)^{l_2}} \exp\big\{-\beta_2(\xi_2 + a_2)\big\} \\
& \times \prod_{i=1}^{k-1} \big(1 + \beta_1 + \beta_1 w_i^{\alpha}\big)^{R_i+1-\delta_i}\ \big(1 + \beta_1 + \beta_1 w_k^{\alpha}\big)^{l_1 - \sum_{j=1}^{k-1}(R_j+1) - \delta_k} \\
& \times \prod_{i=1}^{k-1} \big(1 + \beta_2 + \beta_2 w_i^{\alpha}\big)^{R_i+\delta_i}\ \big(1 + \beta_2 + \beta_2 w_k^{\alpha}\big)^{l_2 - \sum_{j=1}^{k-1}(R_j+1) - 1 + \delta_k} \\
& \times \alpha^{k+b_3-1} \prod_{i=1}^{k} w_i^{\alpha-1} \prod_{i=1}^{k} (1 + w_i^{\alpha}) \exp(-a_3\alpha).
\end{aligned} \tag{16}$$

Let $\phi(\beta_1, \beta_2, \alpha)$ be a function of the model parameters. We are interested in the Bayesian estimator of $\phi(\beta_1, \beta_2, \alpha)$ under the linear-exponential (LINEX) loss function. The LINEX loss function is asymmetric, growing exponentially on one side of zero and approximately linearly on the other, and is therefore useful for problems where overestimation and underestimation have different practical consequences. It is defined as

$$L_{\text{LINEX}}(\hat{\phi}^*, \phi) = \gamma \left[\exp\{c(\hat{\phi}^* - \phi)\} - c(\hat{\phi}^* - \phi) - 1\right], \tag{17}$$

where $\hat{\phi}^*$ is the estimator of ϕ, and $c \neq 0$ and $\gamma > 0$ are known constants. Without loss of generality, we set $\gamma = 1$. The Bayes estimator of $\phi(\beta_1, \beta_2, \alpha)$ under the LINEX loss given in equation (17), denoted by $\hat{\phi}^*$, minimizes the posterior expected loss and is given by the well-known expression

$$\hat{\phi}^* = -\frac{1}{c} \ln \left[E\Big\{\exp\big(-c\,\phi(\beta_1, \beta_2, \alpha)\big) \mid \mathbf{w}, \boldsymbol{\delta}, \mathbf{R}\Big\}\right], \tag{18}$$

where the expectation is taken with respect to the posterior density $\pi(\beta_1, \beta_2, \alpha \mid \mathbf{w}, \boldsymbol{\delta}, \mathbf{R})$ given in equation (16). In the present model, the posterior density given in equation in (16) does not lead to a closed-form solution. Therefore, to obtain approximate Bayesian estimates of β_1, β_2, and α, we employ a Markov chain Monte Carlo (MCMC) scheme, specifically a Gibbs sampler with embedded Metropolis–Hastings (M–H) steps for the conditional distributions that are not of standard form. These posterior samples are then substituted into equation (18) to compute Monte Carlo approximations of the required Bayes estimators under the LINEX loss.

4.1 Gibbs sampling

The Gibbs sampling technique, introduced by Geman and Geman (1984), is used here to obtain the approximate Bayes estimators of the unknown model parameters β_1, β_2, and α. Using the posterior distribution given in equation (16), the full conditional posterior densities of β_1, β_2, and α are obtained as follows:

$$\begin{aligned}\pi_{\beta_1}(\beta_1 \mid \mathbf{w}, \boldsymbol{\delta}, \mathbf{R}, \alpha) \propto{}& \frac{\beta_1^{2k_1+b_1-1}}{(1+\beta_1)^{l_1}} \exp[-\beta_1(\xi_1 + a_1)] \prod_{i=1}^{k-1} (1 + \beta_1 + \beta_1 w_i^{\alpha})^{R_i+1-\delta_i} \\ &\times (1 + \beta_1 + \beta_1 w_k^{\alpha})^{l_1 - \sum_{j=1}^{k-1}(R_j+1) - \delta_k}.\end{aligned} \tag{19}$$

$$\begin{aligned}\pi_{\beta_2}(\beta_2 \mid \mathbf{w}, \boldsymbol{\delta}, \mathbf{R}, \alpha) \propto{}& \frac{\beta_2^{2k_2+b_2-1}}{(1+\beta_2)^{l_2}} \exp[-\beta_2(\xi_2 + a_2)] \prod_{i=1}^{k-1} (1 + \beta_2 + \beta_2 w_i^{\alpha})^{R_i+\delta_i} \\ &\times (1 + \beta_2 + \beta_2 w_k^{\alpha})^{l_2 - \sum_{j=1}^{k-1}(R_j+1) - 1 + \delta_k}.\end{aligned} \tag{20}$$

$$\pi_\alpha(\alpha \mid \mathbf{w}, \boldsymbol{\delta}, \mathbf{R}, \beta_1, \beta_2) \propto \alpha^{k+b_3-1} \prod_{i=1}^{k} w_i^{\alpha-1} \prod_{i=1}^{k}(1+w_i^{\alpha}) \exp(-a_3\alpha) \exp(-\xi_1\beta_1) \exp(-\xi_2\beta_2)$$
$$\times \prod_{i=1}^{k-1} (1+\beta_1+\beta_1 w_i^{\alpha})^{R_i+1-\delta_i} \, (1+\beta_1+\beta_1 w_k^{\alpha})^{l_1-\sum_{j=1}^{k-1}(R_j+1)-\delta_k}$$
$$\times \prod_{i=1}^{k-1} (1+\beta_2+\beta_2 w_i^{\alpha})^{R_i+\delta_i} \, (1+\beta_2+\beta_2 w_k^{\alpha})^{l_2-\sum_{j=1}^{k-1}(R_j+1)-1+\delta_k} . \quad (21)$$

Note that the full conditional posterior distributions in (19)–(21) are not of standard form. Hence, a Metropolis–Hastings (M–H) algorithm is embedded within the Gibbs sampler to generate samples from them.

Algorithm of Gibbs sampler with Metropolis–Hastings

The following steps are used for the generation of posterior MCMC samples:

Step 1. Choose initial values $\beta_1^{(0)}$, $\beta_2^{(0)}$, and $\alpha^{(0)}$.

Step 2. Generate $\alpha^{(j)}$ from (21) using the M–H algorithm:

(i) Propose $\theta \sim N(\ln\alpha^{(j-1)}, 1)$.

(ii) Set $\alpha^{\mathrm{MH}} = \exp(\theta)$.

(iii) Generate $u \sim \mathrm{Unif}(0,1)$.

(iv) Compute

$$\gamma(\alpha^{\mathrm{MH}} \mid \alpha^{(j-1)}) = \min\left\{ \frac{\pi_\alpha(\alpha^{\mathrm{MH}} \mid \text{data})}{\pi_\alpha(\alpha^{(j-1)} \mid \text{data})}, 1 \right\}.$$

(v) If $u \le \gamma$, accept and set $\alpha^{(j)} = \alpha^{\mathrm{MH}}$; otherwise set $\alpha^{(j)} = \alpha^{(j-1)}$.

Step 3. Generate $\beta_1^{(j)}$ from the density given in the equation (19) using an M–H step identical to Step 2.

Step 4. Generate $\beta_2^{(j)}$ from the density given in the equation (20) using an M–H step identical to Step 2.

Step 5. Repeat Steps 2–4 for $j = 1, 2, \ldots, M$ to obtain the MCMC sample

$$(\alpha^{(1)}, \beta_1^{(1)}, \beta_2^{(1)}), \ldots, (\alpha^{(M)}, \beta_1^{(M)}, \beta_2^{(M)}).$$

The proposal distribution used throughout is normal on the log-scale, ensuring positivity of the sampled parameters. Let $c \neq 0$ be the asymmetry parameter. The Bayes estimator under the LINEX loss is

$$\hat{\phi}^* = -\frac{1}{c} \ln(E[\exp(-c\phi) \mid \text{data}])$$

Using the MCMC sample with burn-in M_0, the approximate Bayes estimators are

$$\beta_1^* = -\frac{1}{c}\ln\left[\frac{1}{M-M_0}\sum_{i=M_0+1}^{M}\exp\Big(-c\,\beta_1^{(i)}\Big)\right],$$

$$\beta_2^* = -\frac{1}{c}\ln\left[\frac{1}{M-M_0}\sum_{i=M_0+1}^{M}\exp\Big(-c\,\beta_2^{(i)}\Big)\right],$$

$$\alpha^* = -\frac{1}{c}\ln\left[\frac{1}{M-M_0}\sum_{i=M_0+1}^{M}\exp\Big(-c\,\alpha^{(i)}\Big)\right].$$

4.2 HPD credible intervals

To construct HPD credible intervals, we follow the algorithm of Chen and Shao (1999). Let the post-burn-in MCMC samples be ordered as

$$\beta_{1(1)} \le \cdots \le \beta_{1(M-M_0)}, \qquad \beta_{2(1)} \le \cdots \le \beta_{2(M-M_0)}, \qquad \alpha_{(1)} \le \cdots \le \alpha_{(M-M_0)}.$$

For a nominal confidence level $100(1-\zeta)\%$, the HPD interval for α is

$$\big[\alpha_{(j)},\, \alpha_{(j+h)}\big], \qquad h = \big[(1-\zeta)(M-M_0)\big], \quad j = 1,\ldots,(M-M_0-h),$$

where the chosen j satisfies

$$\alpha_{(j+h)} - \alpha_{(j)} = \min_{1\le i\le (M-M_0-h)} \left[\alpha_{(i+h)} - \alpha_{(i)}\right].$$

Using the same procedure, HPD credible intervals for β_1 and β_2 are obtained analogously.

5 Simulation study

In this section, the performance and efficiency of the proposed estimation methods for two BJPT-II censored power Lindley populations are examined using a Monte Carlo simulation study. Several choices of l_1, l_2, k, and $\mathbf{R}$ are considered, as given in Table 1. In the standard shorthand notation, the censoring plan $\mathbf{R} = (1_{(4)}, 0, (0,1)_{(2)})$ stands for $(R_1 = 1, R_2 = 1, R_3 = 1, R_4 = 1, R_5 = 0, R_6 = 0, R_7 = 1, R_8 = 0, R_9 = 1)$. To compute the approximate Bayes estimates, the prior hyper-parameters are chosen such that $b_1/a_1 = \beta_1$, $b_2/a_2 = \beta_2$, and $b_3/a_3 = \alpha$, i.e. the prior means of the gamma priors coincide with the true parameter values. The performance of the MLEs and Bayesian estimates is assessed in terms of the average absolute bias (AAB) and mean squared error (MSE). For the Bayesian estimates under the LINEX loss function, the asymmetry parameter is fixed at $c = 0.7$. Furthermore, the performance of the asymptotic CIs

Table 1: BJPT-II censoring plans used in the simulation study.

l_2	l_1	k	$\mathbf{R} = (R_1, R_2, \ldots, R_{k-1})$	Censoring scheme (CS) No.
25	25	20	$\mathbf{R} = (0_{(19)})$	1
25	25	20	$\mathbf{R} = ((0,0,1,0,0,0)_{(3)}, 0)$	2
25	25	20	$\mathbf{R} = ((0,0,0)_{(6)}, 1)$	3
25	20	16	$\mathbf{R} = (0_{(15)})$	4
25	20	16	$\mathbf{R} = (0,0,(0,1,0)_{(4)}, 0)$	5
25	20	16	$\mathbf{R} = (1, 0_{(14)})$	6
32	30	26	$\mathbf{R} = (0_{(25)})$	7
32	30	26	$\mathbf{R} = ((0,0,1,0,0,0)_{(4)}, 0)$	8
32	30	26	$\mathbf{R} = ((0,0,0,0)_{(6)}, 1)$	9
30	35	26	$\mathbf{R} = (0_{(25)})$	10
30	35	26	$\mathbf{R} = ((0,0,0,0)_{(6)}, 1)$	11
30	35	26	$\mathbf{R} = (1, (0,0,0,0)_{(6)})$	12
35	35	30	$\mathbf{R} = (0_{(29)})$	13
35	35	30	$\mathbf{R} = ((0,0,0,0,1,0,0)_{(4)}, 0)$	14
35	35	30	$\mathbf{R} = (0_{(27)}, 1, 1)$	15
35	35	32	$\mathbf{R} = (0_{(31)})$	16
35	35	32	$\mathbf{R} = ((0,0,0,0,0,1,0,0,0,0)_{(3)}, 0)$	17
35	35	32	$\mathbf{R} = (0_{(29)}, 0, 1)$	18
35	40	30	$\mathbf{R} = (0_{(29)})$	19
35	40	30	$\mathbf{R} = ((0,0,0,0)_{(7)}, 1)$	20

and HPD credible intervals is summarized via their average lengths (AL) and coverage probabilities (CP).

The Monte Carlo simulation study is carried out for two sets of true parameter values: $(\beta_1, \beta_2, \alpha) = (2, 0.5, 0.5)$ and $(\beta_1, \beta_2, \alpha) = (0.5, 2, 0.5)$, with the corresponding hyper-parameters $(a_1, a_2, a_3, b_1, b_2, b_3) = (2, 60, 75, 4, 30, 37.5)$ and $(a_1, a_2, a_3, b_1, b_2, b_3) = (50, 2.5, 75, 25, 5, 37.5)$, respectively. For the Gibbs sampling procedure, we take $M = 10\,000$ total iterations and a burn-in period of $M_0 = 2\,500$ iterations, i.e. the first M_0 draws are discarded to allow the chain to approach its stationary distribution. The whole simulation experiment is then replicated $N_1 = 10\,000$ times to obtain the AAB and MSE of the point estimators; the corresponding results are presented in Tables 2 and 4. The AL and CP of the interval estimators (asymptotic CIs and HPD credible intervals) are also computed, and the results are given in Tables 3 and 5, respectively. All computations are carried out using the R software environment.

From Tables 2, 4, 3, and 5, the following conclusions can be drawn:

Table 2: The AAB and MSE of the ML and Bayes estimates with $\beta_1 = 2$, $\beta_2 = 0.5$, $\alpha = 0.5$.

CS No.	$\hat{\beta}_1$		$\hat{\beta}_2$		$\hat{\alpha}$		β_1^*		β_2^*		α^*	
	AAB	MSE	AAB	MSE	AAB	MSE	AAB	MSE	AAB	MSE	AAB	MSE
1	0.0399	0.1973	0.0116	0.0147	0.0127	0.0181	0.2684	0.1184	0.0614	0.0041	0.0353	0.0021
2	0.0432	0.1712	0.0139	0.0159	0.0110	0.0121	0.2474	0.0957	0.0663	0.0048	0.0317	0.0015
3	0.0396	0.1654	0.0129	0.0162	0.0107	0.0127	0.2518	0.1055	0.0635	0.0043	0.0371	0.0021
4	0.0642	0.2824	0.0197	0.0209	0.0151	0.0150	0.3126	0.1654	0.0649	0.0046	0.0359	0.0020
5	0.0503	0.1881	0.0212	0.0290	0.0128	0.0150	0.2577	0.1055	0.0860	0.0079	0.0359	0.0020
6	0.0773	0.3613	0.0339	0.0454	0.0207	0.0271	0.2839	0.1318	0.0722	0.0055	0.0304	0.0014
7	0.0374	0.1365	0.0200	0.0271	0.0108	0.0110	0.2597	0.1107	0.0821	0.0071	0.0355	0.0020
8	0.0409	0.1118	0.0263	0.0360	0.0108	0.0102	0.2284	0.0814	0.0933	0.0091	0.0320	0.0016
9	0.0358	0.1151	0.0208	0.0290	0.0092	0.0068	0.2268	0.0824	0.0847	0.0076	0.0379	0.0021
10	0.0334	0.1848	0.0093	0.0131	0.0070	0.0071	0.2341	0.0919	0.0621	0.0041	0.0341	0.0018
11	0.0304	0.1795	0.0099	0.0183	0.0069	0.0087	0.2195	0.0797	0.0641	0.0043	0.0367	0.0018
12	0.0344	0.1807	0.0191	0.0329	0.0094	0.0114	0.2370	0.0964	0.0633	0.0042	0.0307	0.0015
13	0.0333	0.1212	0.0186	0.0245	0.0093	0.0084	0.2406	0.0976	0.0835	0.0074	0.0352	0.0019
14	0.0290	0.0914	0.0188	0.0292	0.0083	0.0088	0.2243	0.0838	0.0858	0.0077	0.0367	0.0021
15	0.0248	0.0818	0.0160	0.0276	0.0070	0.0068	0.2115	0.0719	0.0882	0.0081	0.0377	0.0021
16	0.0333	0.1195	0.0210	0.0322	0.0075	0.0059	0.2476	0.0996	0.0872	0.0079	0.0336	0.0018
17	0.0295	0.0946	0.0201	0.0306	0.0077	0.0072	0.2107	0.0744	0.0911	0.0087	0.0336	0.0017
18	0.0322	0.1126	0.0210	0.0302	0.0089	0.0087	0.2162	0.0802	0.0887	0.0083	0.0360	0.0020
19	0.0332	0.1510	0.0123	0.0143	0.0084	0.0086	0.2292	0.0871	0.0705	0.0053	0.0344	0.0018
20	0.0290	0.1135	0.0117	0.0162	0.0062	0.0056	0.2062	0.0679	0.0722	0.0055	0.0344	0.0018

1) The MLEs and Bayesian estimates of β_1, β_2, and α are mostly unbiased, with relatively small MSEs, as seen in Tables 2 and 4.
2) From Tables 3 and 5, it is observed that the CPs of the asymptotic CIs and HPD credible intervals generally attain the nominal level quite satisfactorily.
3) As the sample sizes (l_1, l_2) and the number of observed failures k increase, the biases of both the MLEs and the Bayesian estimates decrease uniformly.
4) The Bayesian estimates under the LINEX loss function tend to outperform the MLEs in terms of both bias and MSE, owing to the incorporation of prior information.
5) The HPD credible intervals provide shorter interval lengths than the corresponding asymptotic CIs, while maintaining competitive coverage.

Overall, the simulation results indicate that when reliable prior information is available, the Bayesian procedure under LINEX loss can yield substantial gains in terms of MSE and interval length, particularly under highly censored BJPT-II designs. In the absence of such prior information, the MLEs and their asymptotic confidence intervals remain competitive and may be preferred for their simplicity and prior-free interpretation.

Table 3: The AL and CP of 95% asymptotic CIs and HPD credible intervals with $\beta_1 = 2$, $\beta_2 = 0.5$, $\alpha = 0.5$.

CS No.	$\hat{\beta}_1$		$\hat{\beta}_2$		$\hat{\alpha}$		β_1^*		β_2^*		α^*	
	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP
1	1.6475	0.9512	0.7795	0.9619	0.4312	0.9431	1.0862	0.9740	0.2230	0.9870	0.1657	0.9250
2	1.4216	0.9272	0.7094	0.9699	0.4060	0.9699	0.9907	0.9260	0.2216	0.9750	0.1663	0.9490
3	1.5569	0.9304	0.7535	0.9920	0.3967	0.9280	1.0465	0.9610	0.2222	0.9660	0.1634	0.9330
4	2.0293	0.9693	0.7456	0.9816	0.4524	0.9325	1.2460	0.9770	0.2219	0.9680	0.1719	0.9310
5	1.5921	0.9521	0.5687	0.9321	0.4232	0.9214	1.0757	0.9310	0.2197	0.9471	0.1677	0.9180
6	2.0041	0.9593	0.6236	0.9249	0.5091	0.9477	1.1776	0.9680	0.2210	0.9760	0.1791	0.9830
7	1.3782	0.9851	0.5786	0.9355	0.3394	0.9231	0.9747	0.9581	0.2195	0.9852	0.1567	0.9030
8	1.2150	0.9252	0.5080	0.9259	0.3371	0.9456	0.8836	0.9230	0.2192	0.9157	0.1547	0.9336
9	1.3378	0.9412	0.5609	0.9324	0.3292	0.9391	0.9309	0.9670	0.2191	0.9781	0.1538	0.9290
10	1.4786	0.9327	0.7984	0.9651	0.3670	0.9704	0.9571	0.9560	0.2230	0.9751	0.1573	0.9190
11	1.3763	0.9213	0.7569	0.9891	0.3531	0.9239	0.9169	0.9450	0.2222	0.9670	0.1543	0.9275
12	1.4509	0.9646	0.6765	0.9823	0.3816	0.9646	0.9510	0.9600	0.2227	0.9810	0.1624	0.9621
13	1.2731	0.9615	0.5771	0.9385	0.3276	0.9462	0.9027	0.9440	0.2193	0.9790	0.1509	0.9391
14	1.2004	0.9256	0.5503	0.9343	0.3197	0.9339	0.8732	0.9501	0.2190	0.9620	0.1491	0.9520
15	1.2071	0.9612	0.5546	0.9321	0.2987	0.9325	0.8523	0.9570	0.2193	0.9740	0.1472	0.9325
16	1.2153	0.9520	0.5406	0.9451	0.3065	0.9360	0.8898	0.9390	0.2190	0.9721	0.1483	0.9213
17	1.1833	0.9435	0.5388	0.9291	0.3008	0.9435	0.8326	0.9401	0.2194	0.9265	0.1467	0.9423
18	1.2200	0.9462	0.5454	0.9422	0.3053	0.9231	0.8545	0.9641	0.2191	0.9660	0.1465	0.9438
19	1.3706	0.9576	0.6945	0.9765	0.3284	0.9153	0.8975	0.9550	0.2208	0.9780	0.1511	0.9439
20	1.3110	0.9352	0.6784	0.9715	0.3218	0.9537	0.8586	0.9650	0.2207	0.9690	0.1494	0.9437

6 Real data analysis

In this section, we illustrate the proposed inferential procedures using two pairs of real-life data sets. Each pair consists of two independent samples to which the power Lindley distribution (PLD) is fitted. For each pair, we also construct two BJPT-II censored samples according to pre-specified censoring schemes.

6.1 Breakdown duration data

We first consider the breakdown durations of an insulating fluid reported in Nelson (2005) (Chapter 1, Table 1.1). Among the seven voltage levels, the breakdown times at 32 kV and 34 kV are selected and denoted by A_1 and A_2, respectively.

Data set A_1 (32 kV): 0.27, 0.40, 0.69, 0.79, 2.75, 3.91, 9.88, 13.95, 15.93, 27.80, 53.24, 82.85, 89.29, 100.58, 215.10.

Data set A_2 (34 kV): 0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50, 7.35, 8.01, 8.27, 12.06, 31.75, 32.52, 33.91, 36.71, 72.89.

Table 4: The AAB and MSE of the ML and Bayes estimates with $\beta_1 = 0.5$, $\beta_2 = 2$, $\alpha = 0.5$.

CS No.	$\hat{\beta}_1$		$\hat{\beta}_2$		$\hat{\alpha}$		β_1^*		β_2^*		α^*	
	AAB	MSE	AAB	MSE	AAB	MSE	AAB	MSE	AAB	MSE	AAB	MSE
1	0.0110	0.0152	0.0366	0.1932	0.0102	0.0142	0.0716	0.0055	0.2609	0.1124	0.0352	0.0019
2	0.0126	0.0202	0.0347	0.1362	0.0080	0.0102	0.0782	0.0065	0.2358	0.0842	0.0322	0.0016
3	0.0107	0.0132	0.0345	0.1561	0.0092	0.0104	0.0734	0.0058	0.2541	0.1066	0.0374	0.0022
4	0.0179	0.0628	0.0331	0.2480	0.0088	0.0200	0.0501	0.0027	0.2621	0.1117	0.0357	0.0020
5	0.0214	0.0613	0.0431	0.2311	0.0096	0.0121	0.0507	0.0028	0.2466	0.0925	0.0332	0.0017
6	0.0132	0.0342	0.0504	0.5423	0.0123	0.0224	0.0499	0.0026	0.2551	0.1039	0.0292	0.0014
7	0.0136	0.0164	0.0418	0.2089	0.0095	0.0105	0.0782	0.0067	0.2446	0.1020	0.0341	0.0018
8	0.0103	0.0153	0.0341	0.1128	0.0076	0.0072	0.0828	0.0047	0.2129	0.0710	0.0321	0.0016
9	0.0144	0.0184	0.0342	0.1330	0.0090	0.0095	0.0805	0.0069	0.2245	0.1042	0.0365	0.0021
10	0.0295	0.0350	0.0462	0.1329	0.0120	0.0087	0.1044	0.0115	0.2453	0.0992	0.0347	0.0019
11	0.0265	0.0374	0.0413	0.1212	0.0093	0.0067	0.1094	0.0126	0.2328	0.0864	0.0366	0.0020
12	0.0408	0.0768	0.0527	0.2208	0.0137	0.0149	0.1126	0.0131	0.2408	0.0952	0.0318	0.0016
13	0.0168	0.0259	0.0320	0.1241	0.0079	0.0073	0.0956	0.0096	0.2466	0.1008	0.0334	0.0017
14	0.0169	0.0262	0.0301	0.1146	0.0072	0.0065	0.0978	0.0100	0.2189	0.0778	0.0351	0.0019
15	0.0199	0.0268	0.0316	0.0945	0.0083	0.0064	0.0993	0.0104	0.2055	0.0682	0.0366	0.0020
16	0.0175	0.0263	0.0284	0.1083	0.0085	0.0080	0.1004	0.0105	0.2314	0.0937	0.0356	0.0017
17	0.0214	0.0314	0.0314	0.0914	0.0089	0.0091	0.1036	0.0112	0.2086	0.0734	0.0326	0.0016
18	0.0192	0.0274	0.0268	0.0803	0.0077	0.0065	0.1012	0.0108	0.2179	0.0799	0.0342	0.0018
19	0.0240	0.0371	0.0390	0.1670	0.0093	0.0079	0.1143	0.0137	0.2340	0.0912	0.0355	0.0020
20	0.0297	0.0459	0.0364	0.1066	0.0084	0.0052	0.1180	0.0145	0.2161	0.0778	0.0367	0.0021

For the breakdown duration data, the observations represent lifetimes of identical components produced on two parallel production lines tested under the same operating conditions. Since the underlying physical failure mechanism is identical, assuming a common shape parameter α is reasonable, while the scale parameters (β_1, β_2) may differ due to production-line variability. Separate PLD fits are obtained for A_1 and A_2. Figure 2 shows the empirical and fitted pdf curves and the Q–Q plots. Using the `ks.test` function in R, the Kolmogorov–Smirnov distance statistics for A_1 and A_2 are 0.2667 and 0.2631, with associated p-values 0.6604 and 0.5379, respectively. The Q–Q plots show that most sample quantiles lie close to the reference line, with only a few deviations in the upper tail. These observations confirm that the PLD is an appropriate model for both data sets.

Using A_1 and A_2, two BJPT-II censored samples are generated under distinct censoring schemes. These are summarized in Table 6. The MLEs, Bayes estimates, asymptotic confidence intervals, and HPD credible intervals for $(\beta_1, \beta_2, \alpha)$ are displayed in Table 7. Bayesian estimates are obtained under non-informative priors using the Gibbs sampling algorithm with $M = 10{,}000$ and burn-in $M_0 = 2{,}500$. The trace plots in Figure 3 show satisfactory mixing, and the ACF plots indicate negligible autocorrelation. The posterior histograms are nearly symmetric, confirming stable convergence of the Gibbs sampler.

Table 5: The AL and CP of 95% asymptotic CIs and HPD credible intervals with $\beta_1 = 0.5$, $\beta_2 = 2$, $\alpha = 0.5$.

CS No.	$\hat{\beta}_1$		$\hat{\beta}_2$		$\hat{\alpha}$		β_1^*		β_2^*		α^*	
	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP
1	0.7901	0.9612	1.6617	0.9561	0.4145	0.9474	0.2394	0.9680	1.0682	0.9690	0.1648	0.9221
2	0.7188	0.9565	1.4433	0.9271	0.4023	0.9739	0.2381	0.9752	0.9670	0.9412	0.1658	0.9476
3	0.7410	0.9737	1.5178	0.9235	0.3886	0.9386	0.2390	0.9775	1.0383	0.9540	0.1629	0.9265
4	1.2803	0.9721	2.0876	0.9276	0.4961	0.9529	0.2483	0.9850	1.1558	0.9670	0.1695	0.9748
5	1.2147	0.9654	1.6483	0.9307	0.4504	0.9483	0.2474	0.9676	0.9994	0.9206	0.1698	0.9473
6	1.1057	0.9775	2.3518	0.9704	0.5322	0.9508	0.2479	0.9756	1.4415	0.9560	0.1175	0.9620
7	0.7187	0.9821	1.4927	0.9603	0.3967	0.9761	0.2376	0.9675	0.9512	0.9541	0.1564	0.9332
8	0.6681	0.9675	1.2295	0.9182	0.3425	0.9652	0.2368	0.9752	0.8564	0.9320	0.1593	0.9562
9	0.6867	0.9752	1.3578	0.9351	0.3484	0.9675	0.2136	0.9787	0.9165	0.9620	0.1545	0.9263
10	0.5175	0.9125	1.3752	0.9521	0.3442	0.9343	0.2357	0.9212	0.9168	0.9490	0.1564	0.9232
11	0.5014	0.9217	1.3480	0.9597	0.3420	0.9732	0.2347	0.9251	0.9247	0.9641	0.1535	0.9491
12	0.4466	0.9075	1.4180	0.9408	0.3716	0.9342	0.2352	0.9323	0.9363	0.9441	0.1612	0.9540
13	0.5811	0.9310	1.3139	0.9655	0.3290	0.9741	0.2354	0.9512	0.9092	0.9521	0.1516	0.9260
14	0.5684	0.9298	1.2686	0.9649	0.3160	0.9561	0.2352	0.9360	0.8722	0.9630	0.1491	0.9309
15	0.5618	0.9198	1.2170	0.9699	0.3118	0.9323	0.2349	0.9281	0.8378	0.9561	0.1477	0.9260
16	0.5601	0.9231	1.2603	0.9487	0.3138	0.9573	0.2352	0.9313	0.8760	0.9452	0.1479	0.9421
17	0.5327	0.9232	1.1638	0.9308	0.3043	0.9154	0.2351	0.9327	0.8274	0.9439	0.1474	0.9424
18	0.5520	0.9206	1.2213	0.9762	0.2995	0.9444	0.2351	0.9220	0.8536	0.9580	0.1469	0.9471
19	0.4927	0.9259	1.3160	0.9407	0.3240	0.9407	0.2349	0.9190	0.8919	0.9462	0.1509	0.9324
20	0.4669	0.9651	1.2636	0.9384	0.3089	0.9384	0.2349	0.9817	0.8629	0.9490	0.1480	0.9280

Table 6: BJPT-II censored breakdown duration data.

CS No.	k	$\mathbf{R}$	$\mathbf{w}$	$\boldsymbol{\delta}$
CS–1	10	$(0_{(3)}, 1, 0_{(3)}, 2, 0)$	0.19, 0.40, 0.69, 0.79, 3.16, 4.15, 4.67, 4.85, 8.01, 8.27	0, 1, 1, 1, 0, 0, 0, 0, 0, 0
CS–2	12	$(0_{(3)}, 1, 0_{(3)}, 1, 0_{(3)})$	0.19, 0.40, 0.69, 0.79, 3.16, 4.15, 4.67, 4.85, 7.35, 8.01, 8.27, 12.06	0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0

6.2 Failure times data

Next, we consider two data sets from Proschan (1963), with sizes $m = 24$ and $n = 27$, corresponding to failure times (in hours) of the air-conditioning systems installed in 13 Boeing 720 aircraft.

Data set A_3 (Airplane 7914): 3, 5, 5, 13, 14, 15, 22, 22, 23, 30, 36, 39, 44, 46, 50, 72, 79, 88, 97, 102, 139, 188, 197, 210.

Data set A_4 (Airplane 7913): 1, 4, 11, 16, 18, 18, 18, 24, 31, 39, 46, 51, 54, 63, 68,

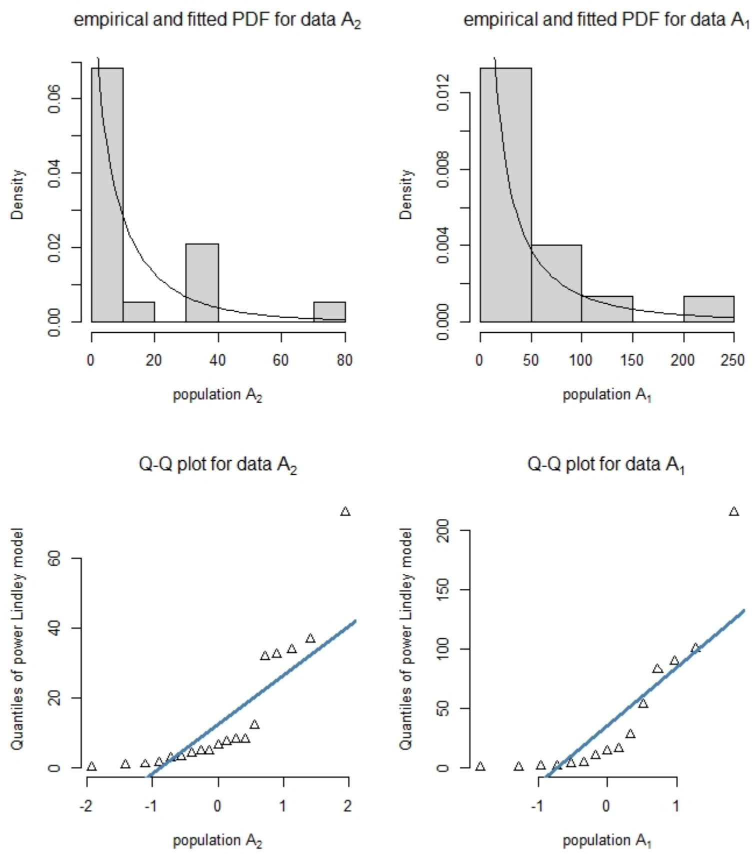


Figure 2: Empirical and fitted pdf curves and Q–Q plots for data sets A_1 and A_2.

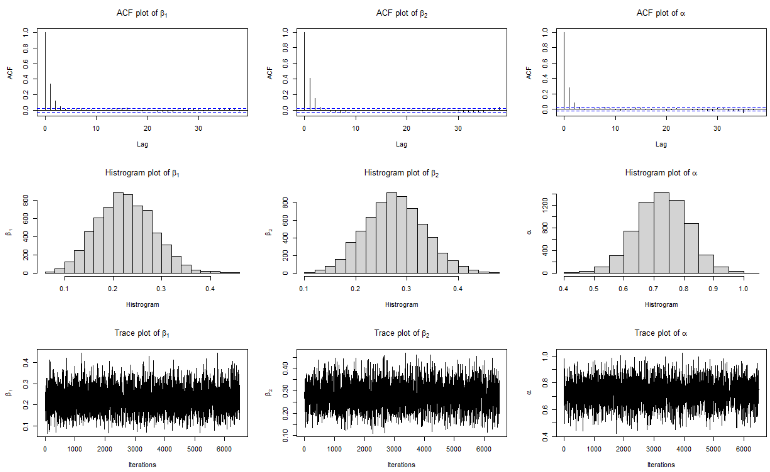
Table 7: Parameter estimates for BJPT-II censored breakdown duration data.

CS No.	Parameter	MLE	Asymptotic CI	Bayes Estimate	HPD CI
CS–1	β_1	0.2338	(0.0413, 0.4264)	0.2235	(0.1171, 0.3415)
	β_2	0.2890	(0.0812, 0.4968)	0.2745	(0.1598, 0.3918)
	α	0.6935	(0.3474, 1.0395)	0.7244	(0.5643, 0.9019)
CS–2	β_1	0.2185	(0.0398, 0.3972)	0.2102	(0.1086, 0.3232)
	β_2	0.3016	(0.0921, 0.5111)	0.2892	(0.1765, 0.4072)
	α	0.6861	(0.3847, 0.9874)	0.7091	(0.5666, 0.8503)

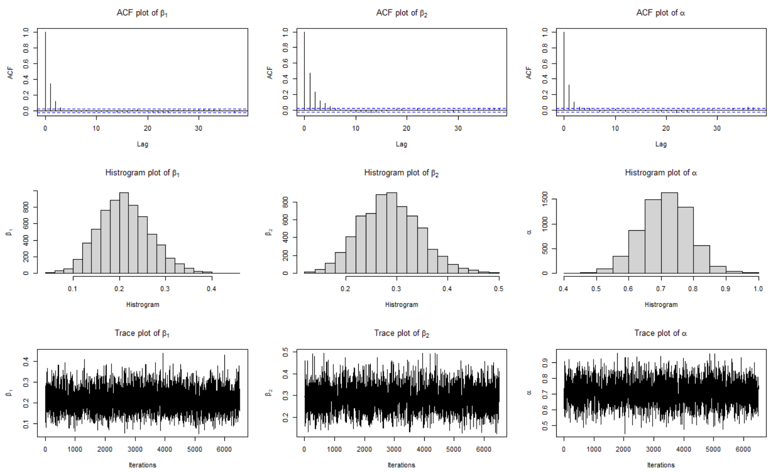
77, 80, 82, 97, 106, 111, 141, 142, 163, 191, 206, 216.

Failure times data corresponds to two design configurations of the same component. As in the previous case, the PLD(β_j, α) model is used with a common shape parameter α and configuration-specific scale parameters β_1 and β_2. Independent preliminary checks using Q–Q plots, fitted densities, and K–S tests indicate that the PLD provides an adequate fit for both data sets. Separate PLD fits are obtained, yielding K–S distance statistics $D = 0.1481$ and $D = 0.2083$, and p-values 0.9284 and 0.6749 for A_3 and A_4, respectively. Figure 4 shows the empirical and fitted pdfs and Q–Q plots.

Using these two data sets, the BJPT-II censored samples summarized in Table 8 are generated under two distinct censoring schemes. The corresponding parameter estimates are presented in Table 9. As in the previous subsection, the trace plots exhibit good mixing, and the ACF plots show negligible autocorrelation. The posterior histograms are nearly symmetric, confirming convergence of the Gibbs sampler for both censoring schemes.



(a) Convergence diagnostics for CS–1



(b) Convergence diagnostics for CS–2

Figure 3: Trace plots, ACF plots, and posterior histograms for breakdown duration data.

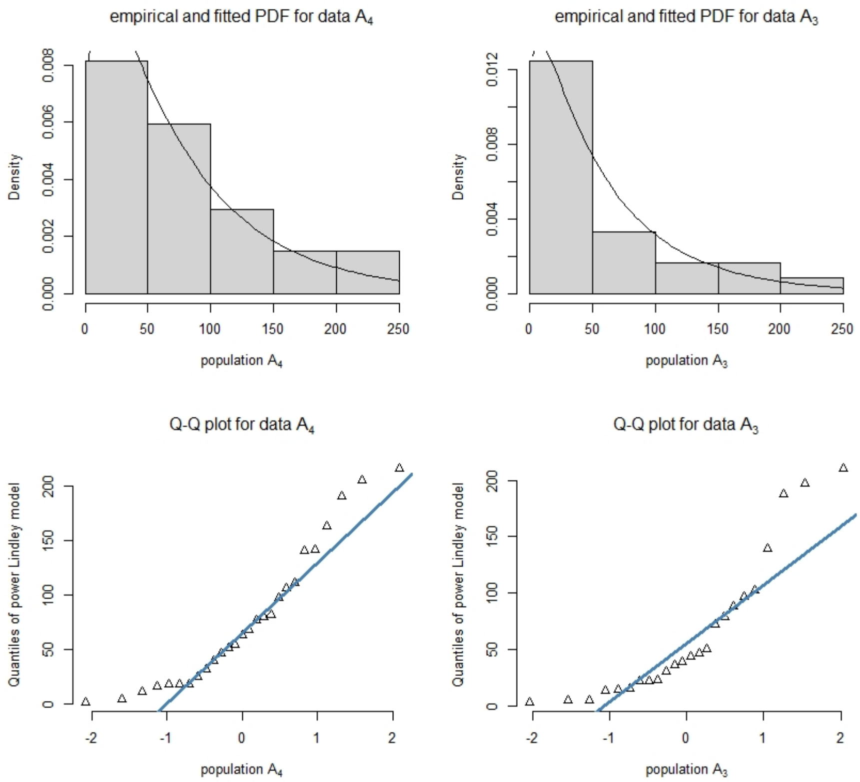


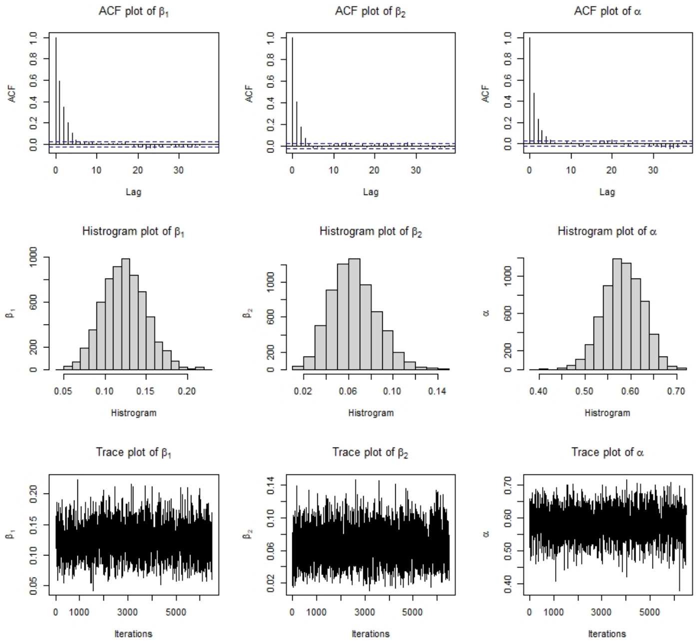
Figure 4: Empirical and fitted pdf curves and Q–Q plots for data sets A_3 and A_4.

Table 8: BJPT-II censored failure times data.

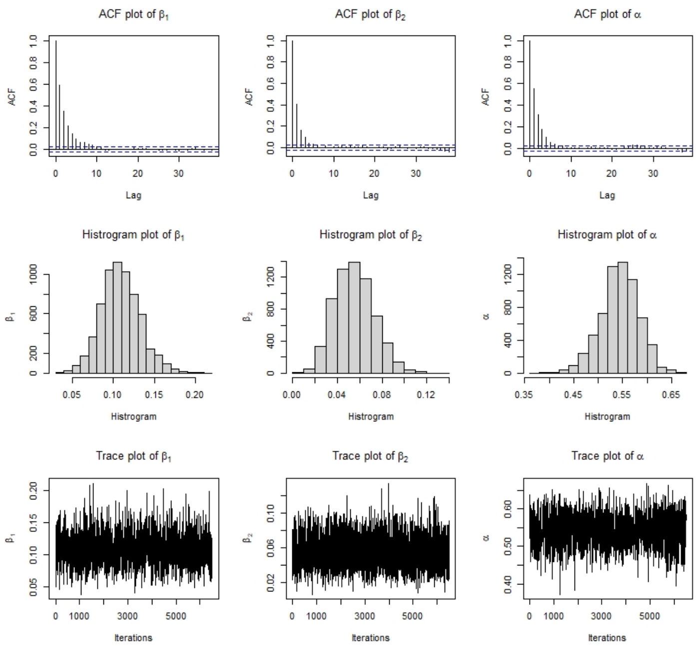
CS No.	k	$\mathbf{R}$	$\mathbf{w}$	$\boldsymbol{\delta}$
CS–3	12	$(0_{(6)}, 1, 1, 1, 1, 1)$	1,4,5,13,14,15,18,23,36,44,50,79	$0, 0, 1_{(10)}$
CS–4	12	$(0_{(4)}, 7, 0_{(4)}, 1, 1)$	1,4,5,13,14,44,46,50,72,82,102,111	$0, 0, 1_{(7)}, 0, 1, 0$

Table 9: Parameter estimates for BJPT-II censored failure times data.

CS No.	Parameter	MLE	Asymptotic CI	Bayes Estimate	HPD CI
CS–3	β_1	0.1406	(0.0117, 0.2696)	0.1332	(0.0742, 0.1885)
	β_2	0.0502	(0.0011, 0.1077)	0.0463	(0.0174, 0.0748)
	α	0.5488	(0.3127, 0.7849)	0.5598	(0.4765, 0.6433)
CS–4	β_1	0.1217	(0.0007, 0.2426)	0.1139	(0.0661, 0.1645)
	β_2	0.0445	(0.0010, 0.0980)	0.0408	(0.0154, 0.0663)
	α	0.5211	(0.2921, 0.7500)	0.5318	(0.4547, 0.6048)



(a) Convergence diagnostics for CS–3



(b) Convergence diagnostics for CS–4

Figure 5: Trace plots, ACF plots, and posterior histograms for failure times data.

7 Optimum censoring plans

In this section, we discuss the choice of an optimal censoring plan for fixed l_1, l_2, k, and a given censoring scheme $\mathbf{R}$. Several authors have investigated optimal censoring designs in reliability literature; see, for example, Bhattacharya et al. (2016), Dube et al. (2016), and Mondal and Kundu (2020). Here, we adopt three standard optimality criteria based on the observed Fisher information matrix under the BJPT-II censoring scheme. Let $I(\hat{\beta}_1, \hat{\beta}_2, \hat{\alpha})$ denote the observed Fisher information matrix, and let $I^{-1}(\hat{\beta}_1, \hat{\beta}_2, \hat{\alpha})$ denote its inverse. For a given censoring plan, we consider the following criteria:

Criterion D (D-optimality): Minimize the determinant of observed variance-covariance matrix $\det\big(I^{-1}(\hat{\beta}_1, \hat{\beta}_2, \hat{\alpha})\big)$.

Criterion A (A-optimality): Minimize the trace of observed variance-covariance matrix $\mathrm{tr}\big(I^{-1}(\hat{\beta}_1, \hat{\beta}_2, \hat{\alpha})\big)$.

Criterion E (E-optimality): Minimize the largest eigenvalue of observed variance-covariance matrix $\lambda_{\max}\big(I^{-1}(\hat{\beta}_1, \hat{\beta}_2, \hat{\alpha})\big)$.

Using the breakdown duration data discussed in Section 6.1, we evaluate these criteria for several candidate BJPT-II censoring plans. The computed values of the A, D, and Ecriteria are listed in Table 10. Among the proposed alternatives, the third censoring plan performs best simultaneously under all three criteria and is therefore considered optimal for this example.

8 Concluding remarks and future scope

In this paper, we studied the classical and Bayesian estimation procedures for BJPT-II censored samples arising from two independent power Lindley populations. Under the classical framework, the maximum likelihood estimators (MLEs) of the model parameters and their asymptotic confidence intervals were derived. Within the Bayesian framework, the posterior estimates were obtained under the LINEX loss function using independent Gamma priors, and the corresponding highest posterior density (HPD) credible intervals were constructed. A numerical simulation study and two real-data applications were carried out to demonstrate the performance and applicability of the proposed methods. Finally, several information-based optimality criteria were applied to determine an optimal censoring plan from a set of candidate BJPT-II schemes.

The simulation study revealed that both the MLEs and Bayes estimators were nearly unbiased across a wide range of censoring levels and parameter configurations. The HPD credible intervals consistently produced shorter interval lengths compared to the asymptotic confidence intervals, indicating their superiority in terms of precision. The real-data analyses further validated the suitability of the power Lindley model with a common shape parameter and distinct scale parameters for modelling the lifetime data from two independent production lines tested under similar operating conditions.

Table 10: Selection of an optimal censoring plan from a set of candidate BJPT-II schemes.

Censoring plan	Criterion A	Criterion D	Criterion E
$k = 12$ $\mathbf{w} = (0.19, 0.40, 0.69, 0.79, 2.75, 3.16, 6.50, 7.35, 8.01, 8.27, 12.06, 31.75)$ $\boldsymbol{\delta} = (0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0)$ $\mathbf{R} = c(\text{rep}(0, 5), 3, \text{rep}(0, 5))$	0.0212	1.0979×10^{-7}	0.0086
$k = 12$ $\mathbf{w} = (0.19, 0.40, 0.69, 0.79, 2.75, 4.15, 4.67, 4.85, 6.50, 7.35, 8.01, 12.06)$ $\boldsymbol{\delta} = (0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0)$ $\mathbf{R} = (0_{(4)}, 1, 0_{(5)}, 1)$	0.0299	1.5952×10^{-7}	0.0184
$k = 12$ $\mathbf{w} = (0.19, 0.69, 0.79, 2.75, 3.16, 4.15, 4.85, 6.50, 7.35, 8.01, 8.27, 31.75)$ $\boldsymbol{\delta} = (0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0)$ $\mathbf{R} = (1, 0_{(4)}, 1, 0_{(4)}, 1)$	0.0211	0.9421×10^{-7}	0.0073
$k = 12$ $\mathbf{w} = (0.19, 0.40, 0.69, 0.79, 2.75, 4.67, 6.50, 7.35, 8.01, 8.27, 12.07, 31.75)$ $\boldsymbol{\delta} = (0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0)$ $\mathbf{R} = (0_{(4)}, 2, 1, 0_{(5)})$	0.0213	1.0553×10^{-7}	0.0088

The analysis highlights the flexibility of the PLD in modelling lifetimes with various ageing behaviours and demonstrates that the BJPT-II censoring scheme can be effectively used to compare the reliability of two production lines under identical testing environments. The simulation results confirm that the proposed estimators perform well under different censoring severities, with informative Bayesian procedures showing notable improvements in highly censored scenarios. The real data examples reinforce that the PLD with a shared shape parameter and line-specific scale parameters is a plausible model for the considered datasets, and that the BJPT-II design yields clear and interpretable comparisons between the two populations.

Possible extensions of this research include: (i) allowing for population-specific shape parameters and examining issues of identifiability and estimation; (ii) incorporating covariates through regression structures on the PLD parameters; (iii) formulating robust or semi-parametric inferential methods that relax the strict PLD assumptions; and (iv) extending the BJPT-II design to more than two populations or adapting it to stress–

strength reliability problems under joint progressive censoring. These areas provide promising directions for future investigation.

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