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The New Topp-Leone-Exponential Generator of Distributions with Applications

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This study introduces a new family of distributions, the Topp–Leone–Exponential generator of distributions. We derive its statistical properties, including the hazard rate function, quantile function, moments, stochastic ordering, order statistics, and Rényi entropy. For parameter estimation, six parameter estimation methods, namely, maximum likelihood, least-squares, weighted least-squares, Cramér-von Mises, Anderson–Darling, and Right-Tail Anderson–Darling are applied, and their consistency is assessed via Monte Carlo simulations for a special case of the new family of distributions. Estimation and application under type I right censoring scheme are presented. Finally, we illustrate the goodness-of-fit of the new family of distributions by fitting a special case to complete and censored data sets from different fields.

keywords: Topp-Leone-G, Generalized Distribution, Estimation Techniques, Simulations.

1 Introduction

Recently, researchers continue to work on extending distributions by using different transformation techniques with the aim of improving their flexibility. Some of the distributions developed using the transformations include Topp-Leone odd log-logistic-G

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family of distributions by Brito et al. (2017). They derived the mathematical properties, developed a TLOLL–Weibull regression model, and demonstrated the flexibility of the distribution in modeling real life data. In another study, Aryal et al. (2017) introduced the Topp-Leone generated Weibull distribution, established several of its statistical and distributional properties, developed a corresponding regression model, provided characterization results, and illustrated its applicability using real-life data sets. Abbas et al. (2017) intoduced the Topp-Leone inverse Weibull distribution, derived its key properties, and showed its usefulness in modeling lifetime data. The Topp–Leone modified Weibull distribution by Alyami et al. (2022) was proposed as a flexible four-parameter lifetime model with various hazard rate shapes and improved performance over several competing distributions in real data applications. Topp-Leone-Marshall-Olkin-G family of distributions was introduced by Chipepa et al. (2020) to enhance model flexibility by combining the Topp–Leone and Marshall–Olkin generators, allowing for a wide range of distributional shapes and improved data fitting performance. The exponentiated Chen distribution was studied by Dey et al. (2017), with emphasis on frequentist parameter estimation, where several estimation methods were compared through Monte Carlo simulations, and the model's applicability was illustrated using real data sets. Rather and Subramanian (2020) analyzed the exponentiated Garima distribution, focusing on frequentist estimation, reliability properties, and real data applications. Elgarhy et al. (2017) proposed the exponentiated Weibull-exponential distribution based on the exponentiated Weibull-G family, studied its properties, estimated parameters using maximum likelihood, and illustrated its usefulness with real data. Nasiru et al. (2019) proposed the exponentiated generalized exponential Dagum distribution, derived its statistical properties, developed maximum likelihood estimators, validated them via simulations and demonstrated the usefulness of the model with real data, Additionally, Warahena-Liyanage et al. (2023) introduced the exponentiated half logistic-Harris-G family of distributions, an extension of the Harris-G family that improves flexibility, tail behavior, and statistical properties. The study derived key distributional characteristics, risk measures such as Value at Tisk (VaR) and Tail Value at Tisk (TVaR), compared multiple parameter estimation methods by simulation, and demonstrated the superiority of the model using real data sets.

Several factors motivated the development of the new family of distributions: i) to define special models with different shapes of hazard rate function; ii) to produce skewness for symmetrical models; iii) to provide consistently better fits than other generalized distributions with the same underlying model; iv) to generalize some existing models in the literature; v) to modulate the weight of the tails of any parental distribution.

The rest of the work is organized in the following manner. Section 2 present the novel Topp-Leone-Exponential generator (TL-Exp-G) of family of distributions, reliability and hazard rate functions, sub-families, linear representation and quantile function. In Section 3, moments, moment generating function, stochastic ordering, the distribution of order statistics, and Rényi entropy are presented. In Section 4, some special models from the TL-Exp-G family of distributions are presented. Risk measures and their simulations are presented in Section 5. Section 6 contains six different estimation methods to estimate the unknown parameters of the TL-Exp-G family of distributions. In Section 7, Monte Carlo simulations are employed to examine the consistency property of six estimation methods for the TL-Exp-G distribution family. Real data applications are given in Section 8. Section 9 is concerned with estimation and application under censoring, followed by some concluding remarks in Section 10.

2 The New Family of Distributions

Al-Nasser and Hanandeh (2025) recently introduced the exponential transformation (ET) family based on the exponential function via a baseline distribution function, say, G with parameter θ, where $\theta \in (0, \infty)$. In this note, we will expand the exponential transformation via the Topp-Leone family of distributions with baseline cumulative distribution function (cdf) $G(x; \varphi)$ for parameter vector φ. The cdf of Exp-generator family of distributions is given by

$$F(x; \alpha, \theta) = G(x; \theta)e^{-\alpha\bar{G}(x;\theta)}, \tag{1}$$

for $x > 0, \theta > 0$ and $\alpha \geq 0$, where $\bar{G}(x; \theta) = 1 - G(x; \theta)$.

In this section, we define the cdf and the probability density function (pdf) of the TL-Exp-G family of distributions. Sub-families are also presented. The cdf and pdf of the TL-Exp-G family of distributions are respectively given by

$$F(x; b, \alpha, \varphi) = \left[1 - \left(1 - G(x; \varphi)e^{-\alpha\bar{G}(x;\varphi)}\right)^2\right]^b \tag{2}$$

and

$$\begin{aligned} f(x; b, \alpha, \varphi) &= 2bg(x; \varphi)e^{-\alpha\bar{G}(x;\varphi)}\left(1 + \alpha G(x; \varphi)\right)\left(1 - G(x; \varphi)e^{-\alpha\bar{G}(x;\varphi)}\right) \\ &\times \left(1 - \left(1 - G(x; \varphi)e^{-\alpha\bar{G}(x;\varphi)}\right)^2\right)^{b-1}, \end{aligned} \tag{3}$$

for $b > 0, \alpha \geq 0$, $\bar{G}(x; \varphi) = 1 - G(x; \varphi)$ and parameter vector φ.

2.1 Sub-Families

Sub-families of the TL-Exp-G family of distributions are given below.

- When $b = 1$, we obtain a new family of distributions with the cdf

$$F(x; \alpha, \varphi) = 1 - \left(1 - G(x; \varphi)e^{-\alpha\bar{G}(x;\varphi)}\right)^2, \tag{4}$$

  for $\alpha \geq 0$ and parameter vector φ.

- When $\alpha = 1$, we obtain a new family of distributions with the cdf

$$F(x;b,\varphi) = \left[1-\left(1-G(x;\varphi)e^{-\bar{G}(x;\varphi)}\right)^2\right]^b, \tag{5}$$

for $b>0$ and parameter vector φ.

- When $b=\alpha=1$, we obtain a new family of distributions with the cdf

$$F(x;\varphi) = 1-\left(1-G(x;\varphi)e^{-\bar{G}(x;\varphi)}\right)^2, \tag{6}$$

for parameter vector φ.

- When $\alpha = 0$, we obtain Topp-Leone-G (TL-G) family of distributions with the cdf

$$F(x;b,\varphi) = \left[1-(1-G(x;\varphi))^2\right]^b, \tag{7}$$

for $b>0$, and parameter vector φ. (See Al-Shomrani et al. (2016) for details).

2.2 Reliability Measures

The survival function and hazard rate function (hrf) of the TL-Exp-G family of distributions are given by

$$S(x;b,\alpha,\varphi) = 1-\left[1-\left(1-G(x;\varphi)e^{-\alpha\bar{G}(x;\varphi)}\right)^2\right]^b, \tag{8}$$

and

$$\begin{aligned} h(x;b,\alpha,\varphi) &= 2bg(x;\varphi)e^{-\alpha\bar{G}(x;\varphi)}\left(1+\alpha G(x;\varphi)\right)\left(1-G(x;\varphi)e^{-\alpha\bar{G}(x;\varphi)}\right) \\ &\times \left(1-\left(1-G(x;\varphi)e^{-\alpha\bar{G}(x;\varphi)}\right)^2\right)^{b-1} \\ &\times \left(1-\left[1-\left(1-G(x;\varphi)e^{-\alpha\bar{G}(x;\varphi)}\right)^2\right]^b\right)^{-1}, \end{aligned} \tag{9}$$

for $b>0, \alpha \geq 0$ and parameter vector φ.

2.3 Linear Representation

In this sub-section, we express the pdf of the TL-Exp-G family of distributions as an infinite linear combination of exponentiated-G (Expo-G) densities. Making use of the generalized series expansions, the pdf of the TL-Exp-G family of distributions can be expressed as follows:

$$f(x;b,\alpha,\varphi) = \sum_{q=0}^{\infty}\xi_{q+1}g_{q+1}(x;\varphi)+\sum_{r=0}^{\infty}w_{r+1}g_{r+1}(x;\varphi),$$

(**see Appendix for derivations**), where $g_{q+1}(x;\varphi) = (q+1)[G(x;\varphi)]^q g(x;\varphi)$ and $g_{r+1}(x;\varphi) = (r+1)[G(x;\varphi)]^r g(x;\varphi)$ are the Expo-G pdf with the power parameters $(q+1)$ and $(r+1)$ respectively, and parameter vector φ, with

$$\begin{aligned} \xi_{q+1} &= 2b \sum_{k,j,l,p=0}^{\infty} (-1)^{k+j+l+p+q} \binom{b-1}{k} \binom{2k+1}{j} \frac{\alpha^l (j+1)^l}{l!(q+1)} \binom{j}{p} \\ &\times \binom{l+p}{q}, \end{aligned} \tag{10}$$

and

$$\begin{aligned} w_{r+1} &= 2b \sum_{k,j,l,p,q=0}^{\infty} (-1)^{k+j+l+p+q} \binom{b-1}{k} \binom{2k+1}{j} \frac{\alpha^l (j+1)^l}{l!(r+1)} \binom{j}{p} \\ &\times \binom{l+p}{q} \binom{q+1}{r}. \end{aligned} \tag{11}$$

Thus, the statistical properties of the TL-Exp-G family of distributions can be obtained from those of the Expo-G family of distributions.

2.4 Quantile Function

The quantile function $Q(u)$ of the TL-Exp-G family of distributions is obtained by solving the non-linear equation

$$F(Q(u)) = \left[1 - \left(1 - G(Q(u))e^{-\alpha\bar{G}(Q(u))}\right)^2\right]^b = u,$$

so that,

$$G(Q(u))e^{-\alpha\bar{G}(Q(u))} = 1 - \left(1 - u^{\frac{1}{b}}\right)^{\frac{1}{2}}. \tag{12}$$

By multiplying both sides of equation (12) by αe^{α}, we have

$$\alpha G(Q(u))e^{\alpha G(Q(u))} = \alpha e^{\alpha} \left[1 - \left(1 - u^{\frac{1}{b}}\right)^{\frac{1}{2}}\right].$$

It can be seen that $\alpha G(Q(u))$ is a Lambert L function of the real argument $\alpha e^{\alpha} \left[1 - \left(1 - u^{\frac{1}{b}}\right)^{\frac{1}{2}}\right]$. The Lambert function say, L, is defined as

$$L(x)e^{L(x)} = x,$$

this has 2 real branches with a branching point located at $(-e^{-1}, 1)$. The lower branch, $L_{-1}(x)$, is defined in the closed interval $[-e^{-1}, 1]$ and it has a negative singularity as

$x \to 0^-$. The upper branch, $L_0(x)$, is defined for $x \in [-e^{-1}, \infty]$.
This implies that,

$$L\left(\alpha e^{\alpha}\left[1-\left(1-u^{\frac{1}{b}}\right)^{\frac{1}{2}}\right]\right) = \alpha G(Q(u)). \tag{13}$$

For any $\alpha > 0$ and $u \in (0,1)$, we have $\alpha G(Q(u)) > 1$ and
$\alpha e^{\alpha}\left[1-\left(1-u^{\frac{1}{b}}\right)^{\frac{1}{2}}\right] < 0$, we can write equation (13) as

$$L_{-1}\left[\alpha e^{\alpha}\left(1-\left(1-u^{\frac{1}{b}}\right)^{\frac{1}{2}}\right)\right] = \alpha G(Q(u)).$$

Hence, the quantile function for the TL-Exp-G family of distributions is given by

$$Q(u) = G^{-1}\left\{\frac{1}{\alpha}L_{-1}\left[\alpha e^{\alpha}\left(1-\left(1-u^{\frac{1}{b}}\right)^{\frac{1}{2}}\right)\right]\right\}. \tag{14}$$

3 Statistical Properties

In this section, we present some statistical properties of the TL-Exp-G family of distributions such as moments, moment generating function, stochastic ordering, distribution of order statistics and Rényi entropy. For convenience and without any loss of generality, we set $f(x;b,\alpha,\varphi) = f(x)$ to be the pdf of the TL-Exp-G family of distributions.

3.1 Moments and Generating Function

Let $Y_{p+1} \sim Exponentiated-G(p+1,\varphi)$, then the k^{th} raw moment, μ'_k of the TL-Exp-G family of distributions is given by

$$\mu'_k = E(X^k) = \int_{-\infty}^{\infty} x^k f(x)dx = \sum_{q=0}^{\infty} \xi_{q+1} E(Y_{p+1}^k) + \sum_{r=0}^{\infty} w_{r+1} E(Y_{p+1}^k),$$

where $E(Y_{p+1}^k)$ is the k^{th} moment of Y_{p+1}, with ξ_{q+1} and w_{r+1} are given by equations (10) and (11), respectively. The moment generating function (MGF), for $|t| < 1$, is given by:

$$M_X(t) = E(e^{tX}) = \sum_{q=0}^{\infty} \xi_{q+1} M_{p+1}(t) + \sum_{r=0}^{\infty} w_{r+1} M_{p+1}(t),$$

where $M_{p+1}(t)$ is the mgf of Y_{p+1}, with ξ_{q+1} and w_{r+1} are given by equations (10) and (11), respectively.

3.2 Stochastic Orderings

This subsection presents some stochastic orders for the TL-Exp-G family of distributions. These include stochastic order, hazard rate order, and likelihood ratio order.

Suppose $F_X(t)$ and $F_Y(t)$ are the cdfs of two random variables X and Y, and define $\bar{F}_X(t) = 1 - F_X(t)$ and $\bar{F}_Y(t) = 1 - F_Y(t)$ as the corresponding survival functions. Then, the random variable X is said to be stochastically smaller than Y if, for all t, $\bar{F}_X(t) \leq \bar{F}_Y(t)$ (or $F_X(t) \geq F_Y(t)$). It is represented by $X <_{st} Y$ or $X \preceq Y$. Moreover, if $\bar{F}_X(t) < \bar{F}_Y(t)$ for some t, then X is stochastically strictly less than Y and denoted as $X \prec Y$. In the case of hazard rate order denoted by $X \preceq_{hr} Y$, $h_X(t) \geq h_Y(t)$ for all t. Similarly, X is said to be smaller than Y in the likelihood ratio order denoted by $X \preceq_{lr} Y$ if $\frac{f_X(t)}{f_Y(t)}$ is decreasing in t. It has been shown that $X \preceq_{lr} Y \implies X \preceq_{hr} Y \implies X \preceq Y$ (Szekli (2012)).

Theorem: Considering two random variables, denoted as X_1 and X_2, which follow the TL-Exp-G family of distributions. Let $X_1 \sim f_1(x; b_1, \alpha, \varphi)$ and $X_2 \sim f_2(x; b_2, \alpha, \varphi)$. If $b_2 > b_1$, then X_1 and X_2 are stochastically ordered.

Proof Note that we can write the pdf's of X_1 and X_2 as follows:

$$\begin{aligned} f_1(x; b_1, \alpha, \varphi) &= 2b_1 g(x;\varphi) e^{-\alpha \bar{G}(x;\varphi)} \left(1 + \alpha G(x;\varphi)\right) \left(1 - G(x;\varphi) e^{-\alpha \bar{G}(x;\varphi)}\right) \\ &\times \left(1 - \left(1 - G(x;\varphi) e^{-\alpha \bar{G}(x;\varphi)}\right)^2\right)^{b_1 - 1} \end{aligned}$$

and

$$\begin{aligned} f_2(x; b_2, \alpha, \varphi) &= 2b_2 g(x;\varphi) e^{-\alpha \bar{G}(x;\varphi)} \left(1 + \alpha G(x;\varphi)\right) \left(1 - G(x;\varphi) e^{-\alpha \bar{G}(x;\varphi)}\right) \\ &\times \left(1 - \left(1 - G(x;\varphi) e^{-\alpha \bar{G}(x;\varphi)}\right)^2\right)^{b_2 - 1}, \end{aligned}$$

respectively. Then, the ratio

$$\frac{f_1(x; b_1, \alpha, \varphi)}{f_2(x; b_2, \alpha, \varphi)} = \frac{b_1}{b_2} \left(1 - \left(1 - G(x;\varphi) e^{-\alpha \bar{G}(x;\varphi)}\right)^2\right)^{b_1 - b_2}. \tag{15}$$

Differentiating equation. (15) with respect to x yields

$$\begin{aligned} \frac{d}{dx}\left(\frac{f_1(x; b_1, \alpha, \varphi)}{f_2(x; b_2, \alpha, \varphi)}\right) &= \frac{(b_1 - b_2)\, b_1}{b_2} \left(1 - \left(1 - G(x;\varphi) e^{-\alpha \bar{G}(x;\varphi)}\right)^2\right)^{b_1 - b_2 - 1} \\ &\times 2g(x;\varphi) e^{-\alpha \bar{G}(x;\varphi)} \left(1 + \alpha G(x;\varphi)\right) \left(1 - G(x;\varphi) e^{-\alpha \bar{G}(x;\varphi)}\right), \end{aligned} \tag{16}$$

which is negative if $b_2 > b_1$. Therefore, likelihood ratio order $X \preceq_{lr} Y$ exists, and we can conclude that the random variables X_1 and X_2 are stochastically ordered.

3.3 Order Statistics

In reliability theory and quality control testing, order statistic plays a vital role in predicting the time to failure of components. Suppose $X_1, X_2, ..., X_z$ are independent and identically distributed random variables from the TL-Exp-G family of distributions. The pdf of the i^{th} order statistic from the TL-Exp-G pdf $f(x)$ can be written as

$$f_{i:z}(x) = \frac{z! f(x)}{(i-1)!(z-i)!} \sum_{s=0}^{z-i} (-1)^s \binom{z-i}{s} [F(x)]^{s+i-1}. \tag{17}$$

Using equations (2) and (3), we have

$$f(x)[F(x)]^{s+i-1} = \sum_{q=0}^{\infty} \psi_{q+1} g_{q+1}(x;\varphi) + \sum_{r=0}^{\infty} \phi_{r+1} g_{r+1}(x;\varphi), \tag{18}$$

(**see Appendix for derivations**), where $g_{q+1}(x;\varphi) = (q+1)[G(x;\varphi)]^q g(x;\varphi)$ and $g_{r+1}(x;\varphi) = (r+1)[G(x;\varphi)]^r g(x;\varphi)$ are the Expo-G pdf with the power parameters $(q+1)$ and $(r+1)$, respectively, and parameter vector φ, with

$$\begin{aligned} \psi_{q+1} &= 2b \sum_{k,j,l,p=0}^{\infty} (-1)^{k+j+l+p+q} \binom{b(s+i)-1}{k} \binom{2k+1}{j} \frac{\alpha^l (j+1)^l}{l!(q+1)} \binom{j}{p} \\ &\times \binom{l+p}{q}, \end{aligned} \tag{19}$$

and

$$\begin{aligned} \phi_{r+1} &= 2b \sum_{k,j,l,p,q=0}^{\infty} (-1)^{k+j+l+p+q} \binom{b(s+i)-1}{k} \binom{2k+1}{j} \frac{\alpha^l (j+1)^l}{l!(r+1)} \binom{j}{p} \\ &\times \binom{l+p}{q} \binom{q+1}{r}. \end{aligned} \tag{20}$$

Thus, by substituting equation (18) into equation (17), the pdf of the i^{th} order statistic from the TL-Exp-G family of distributions can be written as

$$\begin{aligned} f_{i:z}(x) &= \frac{z!}{(i-1)!(z-i)!} \sum_{q=0}^{\infty} \sum_{s=0}^{z-i} (-1)^s \binom{z-i}{s} \psi_{q+1} g_{q+1}(x;\varphi) \\ &+ \frac{z!}{(i-1)!(z-i)!} \sum_{r=0}^{\infty} \sum_{s=0}^{z-i} (-1)^s \binom{z-i}{s} \phi_{r+1} g_{r+1}(x;\varphi). \end{aligned} \tag{21}$$

This shows that the pdf of i^{th} order statistic from the TL-Exp-G family of distributions can be obtained from those of the Expo-G family of distributions.

3.4 Rényi Entropy

Rényi entropy is a measure of randomness or uncertainty in the system. It is mostly used in information theory. Rényi entropy is defined to be

$$I_R(\nu) = \frac{1}{1-\nu}\log\left(\int_0^\infty [f(x)]^\nu dx\right), \nu \neq 1, \nu > 0. \tag{22}$$

Note that

$$\begin{aligned} [f(x)]^\nu &= (2b)^\nu g^\nu(x;\varphi) \sum_{a,k,j,l,p,q=0}^{\infty} (-1)^{k+j+l+p+q} \binom{\nu}{a}\binom{\nu(b-1)}{k}\binom{2k+\nu}{j} \\ &\times \frac{\alpha^{a+l}(j+\nu)^l}{l!}\binom{a+j}{p}\binom{l+p}{q} G^q(x;\varphi). \end{aligned} \tag{23}$$

(**See Appendix for derivations**).
Rényi entropy for the TL-Exp-G family of distributions is given by

$$\begin{aligned} I_R(\nu) &= \frac{1}{1-\nu}\log\Bigg[\sum_{a,k,j,l,p,q=0}^{\infty} (2b)^\nu(-1)^{k+j+l+p+q}\binom{\nu}{a}\binom{\nu(b-1)}{k}\binom{2k+\nu}{j} \\ &\times \frac{\alpha^{a+l}(j+\nu)^l}{l!}\binom{a+j}{p}\binom{l+p}{q}\left[1+\frac{q}{\nu}\right]^{-1} \\ &\times \int_0^\infty \left(\left[1+\frac{q}{\nu}\right](G(x;\varphi))^{\frac{q}{\nu}}(g(x;\varphi))\right)^\nu dx\Bigg] \\ &= \frac{1}{1-\nu}\log\left[\sum_{q=0}^{\infty}\tau_{q+1}\exp\left((1-\theta)I_{REG}\right)\right], \end{aligned} \tag{24}$$

for $\nu > 0$, $\nu \neq 1$, where $I_{REG} = \frac{1}{1-\nu}\log\left(\int_0^\infty \left(\left[1+\frac{q}{\nu}\right](G(x;\varphi))^{\frac{q}{\nu}}(g(x;\varphi))\right)^\nu dx\right)$ is the Rényi entropy of Expo-G distribution with power parameter $(\frac{q}{\nu}+1)$, and

$$\begin{aligned} \tau_{q+1} &= \sum_{a,k,j,l,p=0}^{\infty} (2b)^\nu(-1)^{k+j+l+p+q}\binom{\nu}{a}\binom{\nu(b-1)}{k}\binom{2k+\nu}{j} \\ &\times \frac{\alpha^{a+l}(j+\nu)^l}{l!}\binom{a+j}{p}\binom{l+p}{q}\left[1+\frac{q}{\nu}\right]^{-1}. \end{aligned}$$

Thus, the Rényi entropy of the TL-Exp-G family of distributions can be obtained from that of the Expo-G family of distributions.

4 Some Special Cases

In this section, we introduce three special cases of the TL-Exp-G family of distributions by generalizing the Weibull, Burr XII, and Burr III distributions.

4.1 Topp-Leone-Exponential-Weibull Distribution

If we consider the Weibull distribution with cdf and pdf given by $G(x;\beta) = 1 - e^{-x^{\beta}}$ and $g(x;\beta) = \beta x^{\beta-1} e^{-x^{\beta}}$, respectively, for $\beta > 0$ and $x > 0$, as the baseline distribution, then the Topp-Leone-Exponential-Weibull (TL-Exp-W) distribution has cdf and pdf given by

$$F(x;b,\alpha,\beta) = \left[1 - \left(1 - (1-e^{-x^{\beta}})e^{-\alpha e^{-x^{\beta}}}\right)^{2}\right]^{b} \tag{25}$$

and

$$\begin{aligned} f(x;b,\alpha,\beta) &= 2b\beta x^{\beta-1} e^{-(x^{\beta}+\alpha e^{-x^{\beta}})} \left(1+\alpha(1-e^{-x^{\beta}})\right) \left(1-(1-e^{-x^{\beta}})e^{-\alpha e^{-x^{\beta}}}\right) \\ &\times \left(1 - \left(1-(1-e^{-x^{\beta}})e^{-\alpha e^{-x^{\beta}}}\right)^{2}\right)^{b-1}, \end{aligned} \tag{26}$$

for $b, \beta > 0, \alpha \geq 0$.

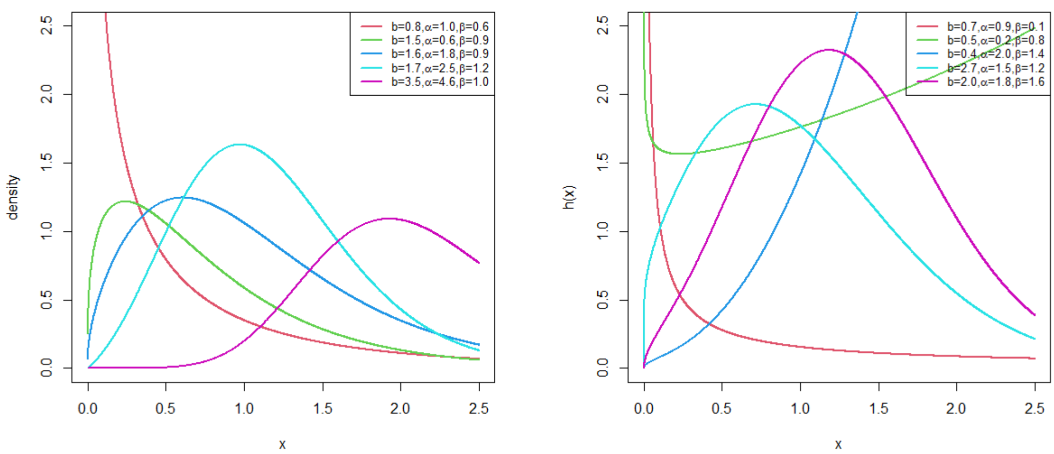

Figure 1: Plots of the pdf and hrf of the TL-Exp-W distribution

Figure 1 shows the plots of the pdf and the hrf of the TL-Exp-W distribution for different parameter values. The pdf can take various shapes including uni-modal, reverse-J, and left or right-skewed. Graphs of the hrf exhibits increasing, decreasing, bathtub, and upside-down bathtub shapes.

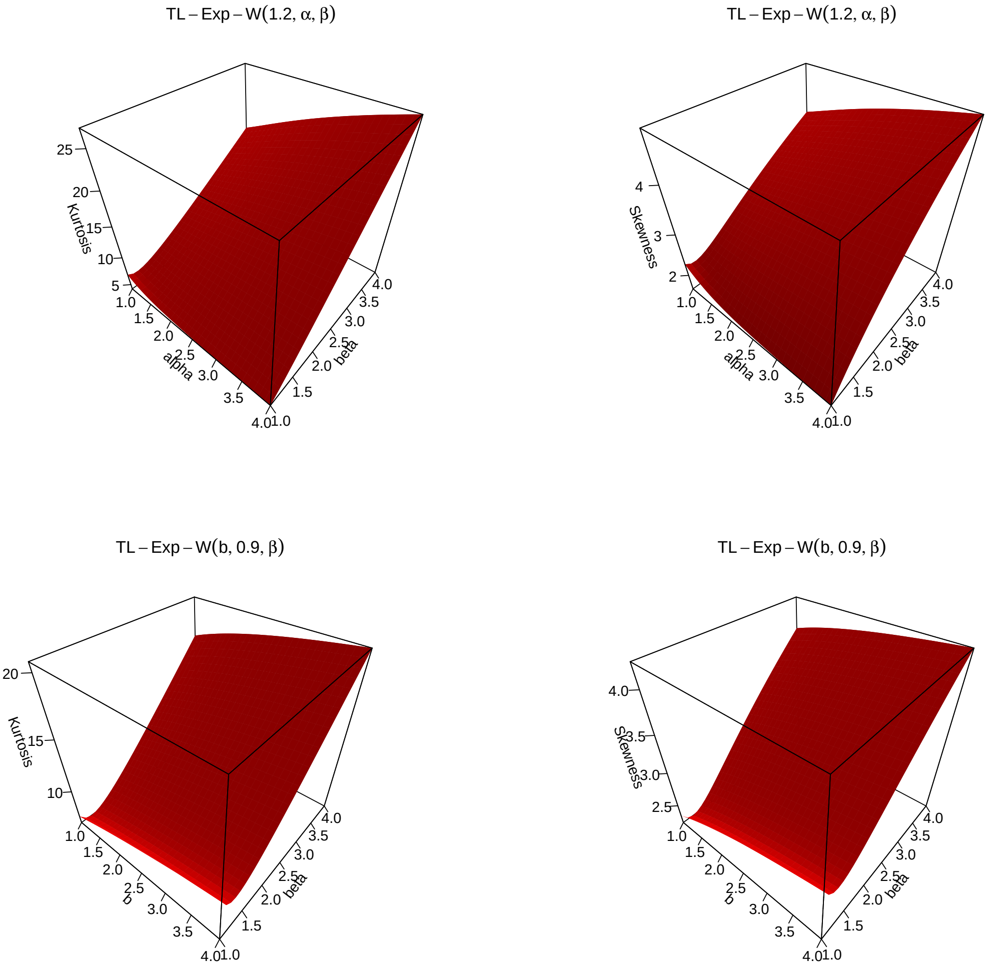

Figure 2: 3D-Plots of the skewness and kurtosis for TL-Exp-W distribution

Figure 2 shows that TL-Exp-W distribution can model data sets with different levels of skewness and kurtosis.

4.2 Topp-Leone-Exponential-Burr XII Distribution

If we consider the baseline distribution as the Burr XII distribution with cdf and pdf given by $G(x; c, k) = 1 - (1 + x^c)^{-k}$ and $g(x; c, k) = ckx^{c-1}(1 + x^c)^{-(k+1)}$, respectively, for $c, k > 0$ and $x > 0$, then the Topp-Leone-Exponential-Burr XII (TL-Exp-BXII) distribution has cdf and pdf given by

$$F(x; b, \alpha, c, k) = \left[1 - \left(1 - (1 - (1 + x^c)^{-k})e^{-\alpha(1+x^c)^{-k}}\right)^2\right]^b$$

and

$$
\begin{aligned}
f(x;b,\alpha,c,k) &= 2bckx^{c-1}(1+x^c)^{-(k+1)}e^{-\alpha(1+x^c)^{-k}}\left(1+\alpha(1-(1+x^c)^{-k})\right)\\
&\times\left(1-\left(1-(1-(1+x^c)^{-k})e^{-\alpha(1+x^c)^{-k}}\right)^2\right)^{b-1}\\
&\times\left(1-(1-(1+x^c)^{-k})e^{-\alpha(1+x^c)^{-k}}\right),
\end{aligned}
$$

for $\alpha \geq 0, b, c, k > 0$. We obtain the new Topp-Leone-Exponential-Log-Logistic (TL-Exp-LLoG) distribution and the new Topp-Leone-Exponential-Lomax (TL-Exp-Lom) distribution when $k = 1$ and $c = 1$, respectively.

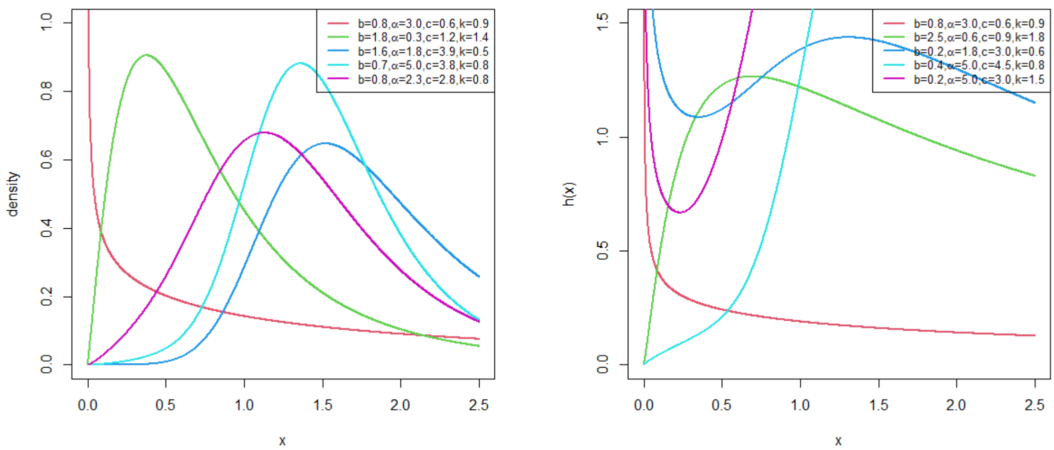

Figure 3: Plots of the pdf and hrf of the TL-Exp-BXII distribution

Figure 3 shows the plots of the pdf and the hrf of the TL-Exp-BXII distribution for different parameter values. The pdf can take various shapes such as uni-modal, reverse-J, left-skewed and right-skewed. Graphs of the hrf exhibit different shapes such as increasing, decreasing, bathtub, bathtub followed by upside-down bathtub, and upside-down bathtub shapes. Figure 4 shows that TL-Exp-BXII distribution can model data sets with different levels of skewness and kurtosis.

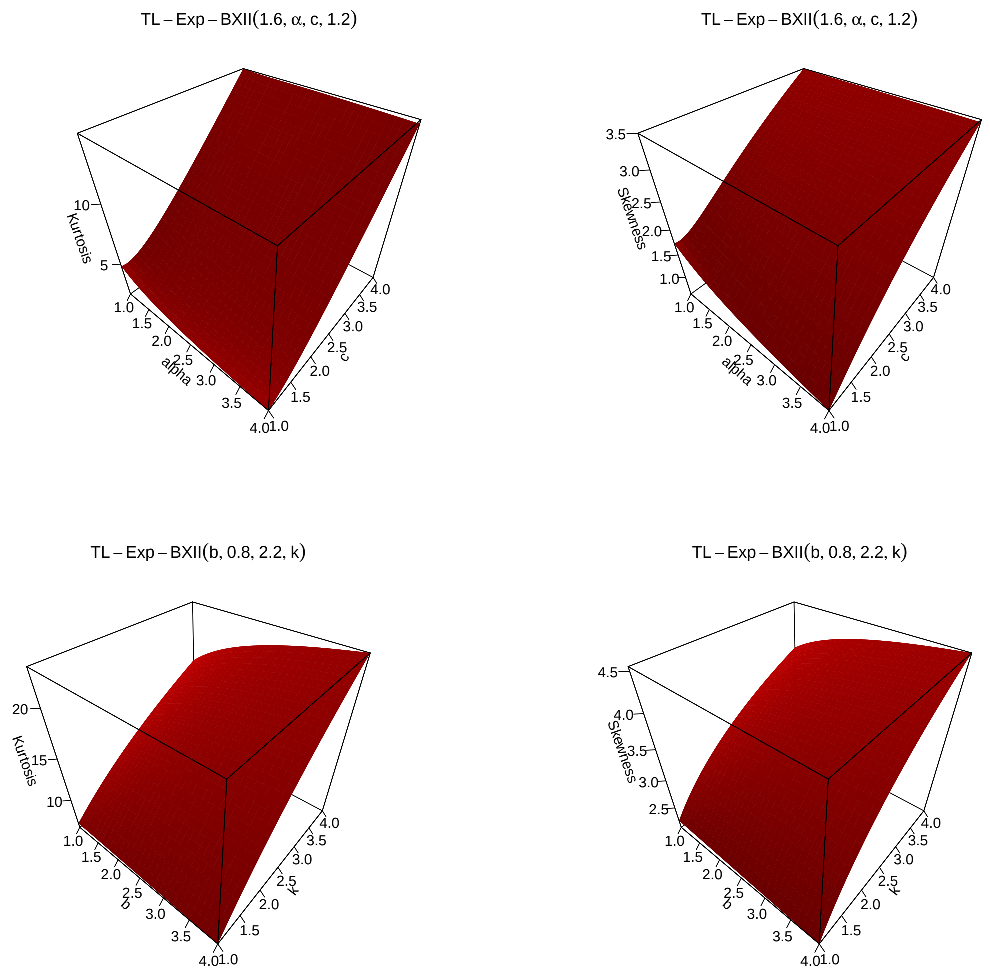

Figure 4: 3D-Plots of the skewness and kurtosis for TL-Exp-BXII distribution

4.3 Topp-Leone-Exponential-Burr III Distribution

If we consider the Burr III distribution with cdf and pdf given by $G(x; c, k) = (1+x^{-c})^{-k}$ and $g(x; c, k) = ckx^{-c-1}(1+x^{-c})^{-k-1}$, respectively, for $c, k > 0$ and $x > 0$, as the baseline distribution, then the Topp-Leone-Exponential-Burr III (TL-Exp-BIII) distribution has cdf and pdf given by

$$F(x; b, \alpha, c, k) = \left[1 - \left(1 - (1 + x^{-c})^{-k} e^{-\alpha(1-(1+x^{-c})^{-k})}\right)^2\right]^b$$

and

$$
\begin{aligned}
f(x;b,\alpha,c,k) &= 2bckx^{-c-1}(1+x^{-c})^{-k-1}e^{-\alpha(1-(1+x^{-c})^{-k})}\left(1+\alpha(1+x^{-c})^{-k}\right) \\
&\times \left(1-\left(1-(1+x^{-c})^{-k}e^{-\alpha(1-(1+x^{-c})^{-k})}\right)^2\right)^{b-1} \\
&\times \left(1-(1+x^{-c})^{-k}e^{-\alpha(1-(1+x^{-c})^{-k})}\right),
\end{aligned}
$$

for $\alpha \geq 0, b, c, k > 0$.

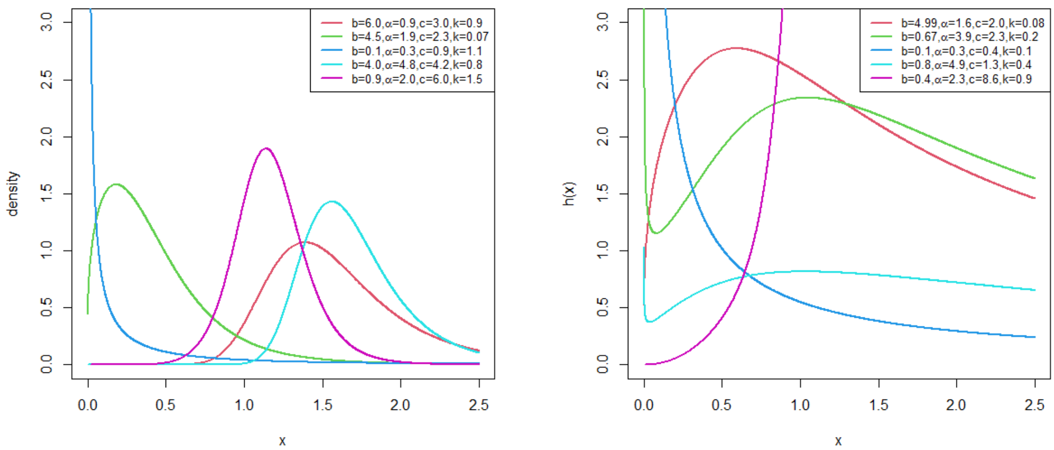

Figure 5: Plots of the pdf and hrf of the TL-Exp-BIII distribution

Figure 5 shows the plots of the pdf and the hrf of the TL-Exp-BIII distribution for different parameter values. The pdf for the TL-Exp-BIII distribution can be reverse-J, uni-modal, right-skewed, and left-skewed. Graphs of the hrf exhibits increasing, decreasing, bathtub, upside-down bathtub and bathtub followed by upside-down bathtub shapes. Figure 6 shows that TL-Exp-BIII distribution can model data sets with different levels of skewness and kurtosis.

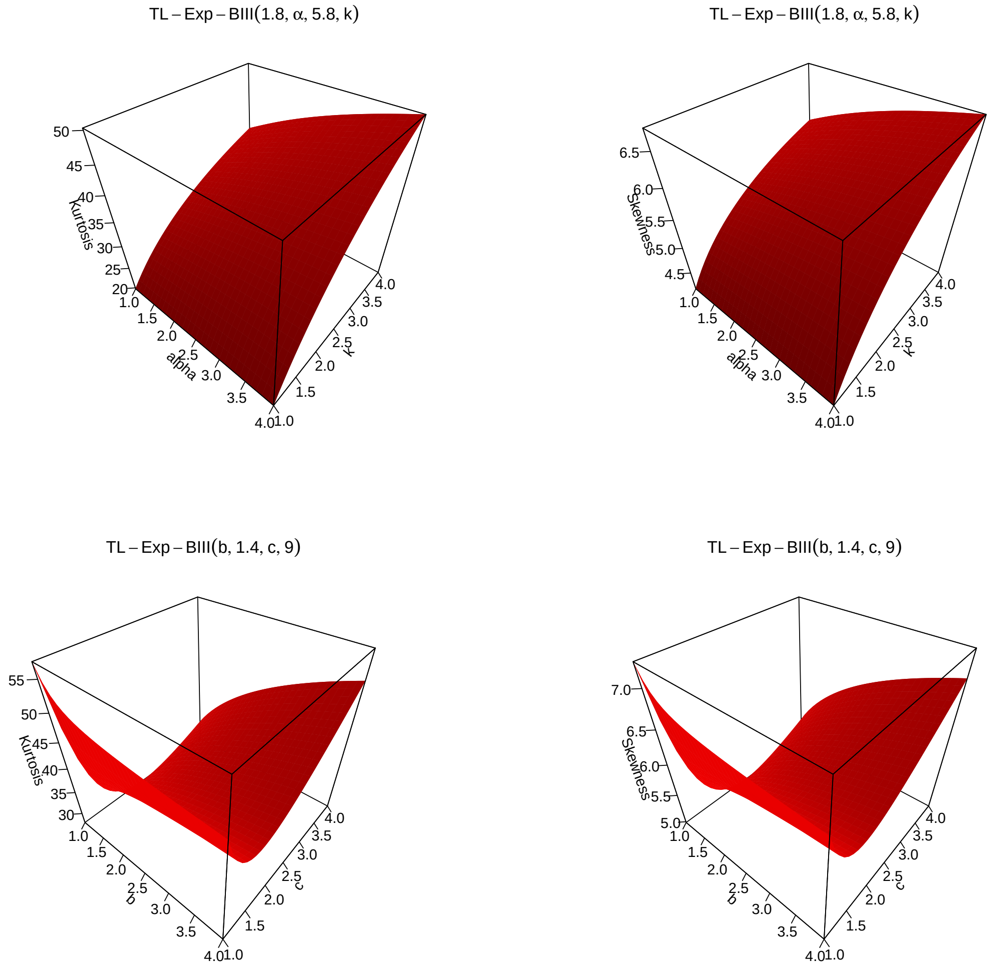

Figure 6: 3D-Plots of the skewness and kurtosis for TL-Exp-BIII distribution

5 Risk Measures and Numerical Studies

In this section, we discuss risk measures including: value at risk (VaR), tail value at risk (TVaR), tail variance (TV), and tail variance premium (TVP) which are usually utilized by financial and actuarial professionals to assess the market risk of a portfolio of instruments.

5.1 Value at Risk

VaR is an actuarial measure that is often used to assess risk in the financial markets. It is referred to as the quantile risk measure or the quantile premium principle, and it is always provided with a stated degree of confidence, such as (90%, 95%, or 99%). The VaR of the TL-Exp-G family of distributions is given by

$$VaR_v = G^{-1}\left\{\frac{1}{\alpha}L_{-1}\left[\alpha e^{\alpha}\left(1-\left(1-v^{\frac{1}{b}}\right)^{\frac{1}{2}}\right)\right]\right\}, \tag{27}$$

where $v \in (0,1)$ is a specified level of significance.

5.2 Tail Value at Risk

TVaR is used to express the expected value of loss in the case that an event beyond the predetermined probability threshold has actually occurred. The TVaR of the TL-Exp-G family of distributions is given as

$$\begin{aligned} TVaR_v &= E(X \mid X > x_v) = \frac{1}{1-v}\int_{VaR_v}^{\infty} xf(x)dx \\ &= \frac{1}{1-v}\left[\sum_{q=0}^{\infty}\int_{VaR_v}^{\infty}\xi_{q+1}g_{q+1}(x;\varphi)dx + \sum_{r=0}^{\infty}\int_{VaR_v}^{\infty}w_{r+1}g_{r+1}(x;\varphi)dx\right], \end{aligned} \tag{28}$$

where ξ_{q+1} and w_{r+1} are given by equations (10) and (11), respectively.

5.3 Tail Variance

Tail variance is the actuarial measure that examines variation outside of the VaR. The TV of the TL-Exp-G family of distributions is given by

$$\begin{aligned} TV_v &= E(X^2 \mid X > x_v) - (TVaR_v)^2 \\ &= \frac{1}{1-v}\int_{VaR_v}^{\infty} x^2 f(x)dx - (TVaR_v)^2 \\ &= \frac{1}{1-v}\left[\sum_{q=0}^{\infty}\xi_{q+1}\int_{VaR_v}^{\infty}x^2 g_{q+1}(x;\varphi)dx + \sum_{r=0}^{\infty}w_{r+1}\int_{VaR_v}^{\infty}x^2 g_{r+1}(x;\varphi)dx\right] - (TVaR_v)^2, \end{aligned} \tag{29}$$

where ξ_{q+1} and w_{r+1} are given by equations (10) and (11), respectively.

5.4 Tail Variance Premium

The TVP is a significant risk measure that is crucial to the study of insurance. The TVP of the TL-Exp-G family distributions is given by

$$TVP_v = TVaR_v + \delta(TV_v), \tag{30}$$

where $0 < \delta < 1$. The TVP of the TL-Exp-G family of distributions can be obtained by substituting equations (28) and (29) into equation (30).

5.5 Numerical Study of Actuarial Measures

In this sub-section, we perform numerical simulation of the risk measures: VaR, TVaR, TV and TVP of the TL-Exp-W distribution and compare to those of the sub-models TL-Exp-W($b = 1, \alpha, \beta$), TL-Exp-W($b, \alpha = 1, \beta$), TL-Exp-W($b = 1, \alpha = 1, \beta$) and Topp-Leone exponential-exponential (TLE-E) by Sanusi et al. (2020) and Topp-Leone inverse Gompertz (TLIG) by Adegoke et al. (2023). Actuarial measures were obtained by generating a random sample of size $n = 100$ from these distributions, and using the maximum likelihood method of estimation to estimate the model parameters. Secondly, a repetition of 1000 iterations are made in order to find the values of the risk measures for the distributions. The model with higher values of VaR, TVaR, TV and TVP is said to have a heavier tail. From the figures in Table 1, we conclude that the TL-Exp-W distribution have a heavier tail than the models including the TLE-E and TLIG distributions, hence it is suitable for modelling heavy-tailed data.

Table 1: Simulation Results of VaR, TVaR, TV and TVP

Significance level		0.7	0.75	0.8	0.85	0.9	0.95
TL-Exp-W($b = 2.5, \alpha = 0.9, \beta = 1.8$)	VaR	0.8272	0.9483	1.0970	1.2897	1.5638	2.0401
	TVaR	1.5068	1.6310	1.7838	1.9821	2.2646	2.7554
	TV	0.5078	0.5143	0.5226	0.5337	0.5497	0.5783
	TVP	1.8622	2.0167	2.2019	2.4358	2.7594	3.3048
TL-Exp-W($b = 1, \alpha = 0.9, \beta = 1.8$)	VaR	0.2840	0.3161	0.3530	0.3971	0.4539	0.5392
	TVaR	0.3983	0.4180	0.4390	0.4606	0.4789	0.4651
	TV	0.0157	0.0164	0.0182	0.0222	0.0316	0.0572
	TVP	0.4093	0.4304	0.4536	0.4795	0.5074	0.5194
TL-Exp-W($b = 2.5, \alpha = 1, \beta = 1.8$)	VaR	0.3866	0.4136	0.4443	0.4808	0.5276	0.5983
	TVaR	0.4928	0.5114	0.5321	0.5556	0.5818	0.6039
	TV	0.0118	0.0121	0.0129	0.0150	0.0202	0.0380
	TVP	0.5011	0.5205	0.5425	0.5683	0.6000	0.6400
TL-Exp-W($b = 1, \alpha = 1, \beta = 1.8$)	VaR	0.2608	0.2914	0.3268	0.3692	0.4240	0.5067
	TVaR	0.3801	0.4009	0.4240	0.4496	0.4768	0.4919
	TV	0.0136	0.0137	0.0144	0.0165	0.0223	0.0417
	TVP	0.3896	0.4112	0.4356	0.4637	0.4968	0.5315
TLE-E($\sigma = 2.5, \lambda = 0.9, \gamma = 1.8$)	VaR	0.4165	0.4446	0.4764	0.5142	0.5626	0.6357
	TVaR	0.5161	0.5332	0.5515	0.5705	0.5871	0.5783
	TV	0.0151	0.0163	0.0187	0.0234	0.0337	0.0627
	TVP	0.5267	0.5455	0.5665	0.5904	0.6174	0.6380
TLIG($\alpha = 2.5, \lambda = 0.9, \beta = 1.8$)	VaR	0.6455	0.5878	0.5254	0.4559	0.3744	0.2681
	TVaR	0.0003	0.0004	0.0005	0.0007	0.0011	0.0021
	TV	0.0002	0.0003	0.0004	0.0006	0.0008	0.0016
	TVP	0.0006	0.0007	0.0009	0.0012	0.0018	0.0037

6 Parameter Estimation

In statistical analysis, estimating the unknown parameters for the given sample is an important concept. In this section, we estimate the unknown parameters of the TL-Exp-G family of distributions by using six different estimation methods, namely, Maximum Likelihood (MLE), Anderson-Darling (ADE), Right-Tail Anderson-Darling (RADE), Ordinary Least Squares (LS), Weighted Least Squares (WLS) and Cramér-von Mises

(CVME).

6.1 Maximum Likelihood Estimation

Let $X \sim TL-Exp-G(b,\alpha,\varphi)$ and $\Delta=(b,\alpha,\varphi)^T$ be the vector of model parameters, then the log-likelihood function $\ell_n=\ell_n(\Delta)$ based on a random sample of size n from the TL-Exp-G family of distributions is given by

$$\begin{aligned}\ell_n(\Delta) &= (n)\ln(2b)-\alpha\sum_{i=1}^{n}\bar{G}(x_i;\varphi)+\sum_{i=1}^{n}\ln\left(1+\alpha G(x_i;\varphi)\right)\\&+\sum_{i=1}^{n}\ln\left(1-G(x_i;\varphi)e^{-\alpha\bar{G}(x_i;\varphi)}\right)+\sum_{i=1}^{n}\ln\left(g(x_i;\varphi)\right)\\&+(b-1)\sum_{i=1}^{n}\ln\left(1-\left(1-G(x_i;\varphi)e^{-\alpha\bar{G}(x_i;\varphi)}\right)^2\right).\end{aligned}$$

In order to obtain the estimates of the unknown parameters from the TL-Exp-G family of distributions, we solve $U=(\frac{\partial\ell_n}{\partial b},\frac{\partial\ell_n}{\partial\alpha},\frac{\partial\ell_n}{\partial\varphi_k})^T=\mathbf{0}$, using a numerical method such as Newton-Raphson procedure. Elements of the score vector are given in the appendix.

The Fisher information matrix is given by

$$J(\Delta)=\begin{pmatrix}J_{b,b}(\Delta) & J_{b,\alpha}(\Delta) & J_{b,\psi}(\Delta)\\ J_{\alpha,b}(\Delta) & J_{\alpha,\alpha}(\Delta) & J_{\alpha,\psi}(\Delta)\\ J_{\psi,b}(\Delta) & J_{\psi,\alpha}(\Delta) & J_{\psi,\psi}(\Delta)\end{pmatrix}, \tag{31}$$

where $J_{i,j}=-\frac{\partial^2\ell_n(\boldsymbol{\Delta})}{\partial_i\partial_j}$ for $i,j=b,\alpha,\psi$.

The multivariate normal distribution $N_{q+2}(\underline{0},J(\hat{\Delta})^{-1})$, where the mean vector $\underline{\mathbf{0}}=(0,0,\underline{0})^T$ and $J(\hat{\Delta})^{-1}$ is the observed Fisher information matrix evaluated at $\hat{\Delta}$, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions.

6.2 Anderson-Darling Estimation

Suppose $X_{(1)},X_{(2)},...,X_{(n)}$ are the order statistics of a random sample of size n from the TL-Exp-G family of distributions. Then, the Anderson-Darling estimates (ADEs) of the TL-Exp-G family of distributions are obtained by minimizing the following function:

$$A(b,\alpha,\varphi)=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log\left(F(x_{(i)};b,\alpha,\varphi)\right)+\log\left(S(x_{(i)};b,\alpha,\varphi)\right)\right],$$

where $F(x_{(i)};b,\alpha,\varphi)$ and $S(x_{(i)};b,\alpha,\varphi)$ be the cdf and survival function of the i^{th} order statistic from the TL-Exp-G family of distributions. The ADEs can also be derived by solving the non-linear equations

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\vartheta_z\left(x_{(i)};b,\alpha,\varphi\right)}{F(x_{(i)};b,\alpha,\varphi)}-\frac{\vartheta_z\left(x_{(n+1-i)};b,\alpha,\varphi\right)}{S\left(x_{(n+1-i)};b,\alpha,\varphi\right)}\right]=0, z=1,2,3, \tag{32}$$

where

$$\begin{aligned}
\vartheta_1\left(x_{(i)};b,\alpha,\varphi\right) &= \frac{\partial F\left(x_{(i)};b,\alpha,\varphi\right)}{\partial b},\\
\vartheta_2\left(x_{(i)};b,\alpha,\varphi\right) &= \frac{\partial F\left(x_{(i)};b,\alpha,\varphi\right)}{\partial \alpha},\\
\text{and}\quad \vartheta_3\left(x_{(i)};b,\alpha,\varphi\right) &= \frac{\partial F\left(x_{(i)};b,\alpha,\varphi\right)}{\partial \varphi_j},
\end{aligned} \tag{33}$$

$j=1,2,.....,k.$

6.3 Right-Tail Anderson-Darling Estimation

Right-Tail Anderson-Darling estimates (RADEs) of the TL-Exp-G family of distributions are determined by minimizing

$$R(b,\alpha,\varphi)=\frac{n}{2}-2\sum_{i=1}^{n}F(x_{(i)};b,\alpha,\varphi)-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\log S(x_{(n-i+1)};b,\alpha,\varphi).$$

The RADEs may also be obtained by solving the non-linear equation

$$-2\sum_{i=1}^{n}\frac{\vartheta_z\left(x_{(i)};b,\alpha,\varphi\right)}{F(x_{(i)};b,\alpha,\varphi)}+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\,\frac{\vartheta_z\left(x_{(i)};b,\alpha,\varphi\right)}{S(x_{n+1-i:n};b,\alpha,\varphi)}=0,$$

where $\vartheta_z\left(x_{(i)};b,\alpha,\varphi\right)$ are defined in equation (33).

6.4 Ordinary Least Squares Estimation

The ordinary least squares estimates (OLSEs) of the parameters of the TL-Exp-G family of distributions are obtained by minimizing the function

$$V\left(b,\alpha,\varphi\right)=\sum_{i=1}^{n}\left[F\left(x_{(i)};b,\alpha,\varphi\right)-\frac{i}{n+1}\right]^2.$$

The OLSEs can be obtained by solving the non-linear equations

$$\sum_{i=1}^{n}\left[F\left(x_{(i)};b,\alpha,\varphi\right)-\frac{i}{n+1}\right]\vartheta_z\left(x_{(i)};b,\alpha,\varphi\right)=0, z=1,2,3,$$

where $\vartheta_z\left(x_{(i)};b,\alpha,\varphi\right)$ are defined in equation (33).

6.5 Weighted Least Squares Estimation

The weighted least squares estimates (WLSEs) of the parameters of the TL-Exp-G family of distributions are obtained by minimizing the function

$$W(b,\alpha,\varphi)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F\left(x_{(i)};b,\alpha,\varphi\right)-\frac{i}{n+1}\right]^2$$

with respect to b,α and baseline parameter vector φ. The WLSEs can be obtained by solving the non-linear equations

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F\left(x_{(i)};b,\alpha,\varphi\right)-\frac{i}{n+1}\right]\vartheta_z\left(x_{(i)};b,\alpha,\varphi\right)=0, z=1,2,3,$$

where $\vartheta_z\left(x_{(i)};b,\alpha,\varphi\right)$ are defined in equation (33).

6.6 Cramér-von Mises Estimation

The Cramér-von Mises estimates (CVMEs) of the parameters of the TL-Exp-G family of distributions parameters are obtained through the minimization of the function

$$C(b,\alpha,\varphi)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F\left(x_{(i)};b,\alpha,\varphi\right)-\frac{2i-1}{2n}\right]^2$$

with respect to b,α and baseline parameter vector φ. The CVMEs can also be obtained by solving the non-linear equations

$$\sum_{i=1}^{n}\left[F\left(x_{(i)};b,\alpha,\varphi\right)-\frac{2i-1}{2n}\right]\vartheta_z\left(x_{(i)};b,\alpha,\varphi\right)=0, z=1,2,3,$$

where $\vartheta_z\left(x_{(i)};b,\alpha,\varphi\right)$ are defined in equation (33).

7 Simulations

In this section, a Monte Carlo simulation study is used to assess the performance of parameter estimates of the TL-Exp-W distribution obtained via six estimation methods discussed in Section 5. A simulation study was carried out by generating $N=3000$ random samples from the TL-Exp-W distribution for various sample sizes of $n=25,50,100,200,400,$ and 800.

To assess performance of the different estimation methods, we used the statistics: average bias (ABias) and root mean square error (RMSE). The ABias and RMSE for the estimated parameter, say, $\hat{\lambda}$, are given by:

$$ABias(\hat{\lambda})=\frac{1}{N}\sum_{i=1}^{N}(\hat{\lambda}_i-\lambda),\quad \text{and}\quad RMSE(\hat{\lambda})=\sqrt{\frac{\sum_{i=1}^{N}(\hat{\lambda}_i-\lambda)^2}{N}},$$

respectively.

Table 2: Simulation Results for Different Estimation Methods for $b = 2.0, \alpha = 0.5, \beta = 0.5$

		MLE		LS		WLS		RADE		CVME		ADE	
n	Parameter	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE
25	b	$0.2683^{\{1\}}$	$1.1715^{\{3\}}$	$-1.9808^{\{6\}}$	$2.3035^{\{6\}}$	$-0.9153^{\{3\}}$	$1.1324^{\{2\}}$	$-0.5628^{\{2\}}$	$0.9784^{\{1\}}$	$-1.3668^{\{4\}}$	$1.3989^{\{4\}}$	$-1.7299^{\{5\}}$	$1.8985^{\{5\}}$
	α	$0.0924^{\{1\}}$	$0.6355^{\{2\}}$	$0.5207^{\{4\}}$	$1.0629^{\{6\}}$	$-0.5656^{\{5\}}$	$0.7910^{\{3\}}$	$-0.6670^{\{6\}}$	$0.9475^{\{5\}}$	$0.2935^{\{3\}}$	$0.8003^{\{4\}}$	$-0.2866^{\{2\}}$	$0.3234^{\{1\}}$
	β	$0.0706^{\{1\}}$	$0.2730^{\{2\}}$	$1.3680^{\{5\}}$	$1.4282^{\{5\}}$	$-0.1744^{\{2\}}$	$0.1765^{\{1\}}$	$-0.7021^{\{4\}}$	$0.8402^{\{4\}}$	$1.3814^{\{6\}}$	$1.4655^{\{6\}}$	$0.5918^{\{3\}}$	$0.6794^{\{3\}}$
	$\sum$ ranks	10		32		16		22		27		19	
50	b	$0.1791^{\{1\}}$	$0.9308^{\{2\}}$	$-1.4391^{\{6\}}$	$1.7747^{\{6\}}$	$-0.7864^{\{3\}}$	$0.9677^{\{4\}}$	$-0.5333^{\{2\}}$	$0.9414^{\{3\}}$	$-1.2674^{\{5\}}$	$1.3551^{\{5\}}$	$-0.7968^{\{4\}}$	$0.9165^{\{1\}}$
	α	$0.0731^{\{1\}}$	$0.4771^{\{2\}}$	$0.2888^{\{4\}}$	$0.9013^{\{6\}}$	$-0.4637^{\{5\}}$	$0.4792^{\{3\}}$	$-0.4936^{\{6\}}$	$0.6645^{\{5\}}$	$0.2600^{\{2\}}$	$0.6316^{\{4\}}$	$-0.2828^{\{3\}}$	$0.3081^{\{1\}}$
	β	$0.0309^{\{1\}}$	$0.2730^{\{2\}}$	$1.367^{\{6\}}$	$1.4222^{\{5\}}$	$-0.1488^{\{2\}}$	$0.1502^{\{1\}}$	$-0.5538^{\{3\}}$	$0.6612^{\{3\}}$	$1.3137^{\{5\}}$	$1.4729^{\{6\}}$	$0.5662^{\{4\}}$	$0.6692^{\{4\}}$
	$\sum$ ranks	9		33		18		22		27		17	
100	b	$0.1025^{\{1\}}$	$0.73905^{\{1\}}$	$-1.4297^{\{6\}}$	$1.4530^{\{6\}}$	$-0.4143^{\{4\}}$	$0.7523^{\{3\}}$	$-0.3987^{\{3\}}$	$0.7401^{\{2\}}$	$-1.1514^{\{2\}}$	$1.2671^{\{5\}}$	$-0.7726^{\{5\}}$	$0.8534^{\{4\}}$
	α	$0.0362^{\{1\}}$	$0.3402^{\{2\}}$	$0.2789^{\{4\}}$	$0.6442^{\{6\}}$	$-0.3062^{\{5\}}$	$0.3441^{\{3\}}$	$-0.4839^{\{6\}}$	$0.5037^{\{4\}}$	$0.2586^{\{2\}}$	$0.6065^{\{5\}}$	$-0.2788^{\{3\}}$	$0.2911^{\{1\}}$
	β	$0.0160^{\{1\}}$	$0.1012^{\{1\}}$	$1.2365^{\{6\}}$	$1.2903^{\{6\}}$	$-0.1371^{\{2\}}$	$0.1383^{\{2\}}$	$-0.5281^{\{3\}}$	$0.6260^{\{3\}}$	$1.2285^{\{5\}}$	$1.2718^{\{5\}}$	$0.5765^{\{4\}}$	$0.6538^{\{4\}}$
	$\sum$ ranks	7		34		19		21		24		21	
200	b	$0.0669^{\{1\}}$	$0.5730^{\{1\}}$	$-1.3903^{\{6\}}$	$1.4014^{\{6\}}$	$-0.3876^{\{3\}}$	$0.6833^{\{3\}}$	$-0.3487^{\{2\}}$	$0.6169^{\{2\}}$	$-1.0774^{\{5\}}$	$1.1765^{\{5\}}$	$-0.7369^{\{4\}}$	$0.7901^{\{4\}}$
	α	$0.0252^{\{1\}}$	$0.2557^{\{1\}}$	$0.2595^{\{4\}}$	$0.3250^{\{5\}}$	$-0.1075^{\{2\}}$	$0.2590^{\{2\}}$	$-0.4158^{\{6\}}$	$0.4245^{\{6\}}$	$0.2562^{\{3\}}$	$0.3098^{\{4\}}$	$-0.2610^{\{5\}}$	$0.2705^{\{3\}}$
	β	$0.0058^{\{1\}}$	$0.0686^{\{1\}}$	$1.1022^{\{6\}}$	$1.1299^{\{6\}}$	$-0.1135^{\{2\}}$	$0.1240^{\{2\}}$	$-0.5203^{\{3\}}$	$0.6126^{\{4\}}$	$1.0900^{\{5\}}$	$1.1115^{\{5\}}$	$0.5354^{\{4\}}$	$0.5703^{\{3\}}$
	$\sum$ ranks	6		33		14		23		27		23	
400	b	$0.0427^{\{1\}}$	$0.3158^{\{1\}}$	$-1.1166^{\{6\}}$	$1.3008^{\{6\}}$	$-0.1040^{\{2\}}$	$0.5962^{\{3\}}$	$-0.3455^{\{3\}}$	$0.5067^{\{2\}}$	$-1.0615^{\{5\}}$	$1.1707^{\{5\}}$	$-0.7009^{\{4\}}$	$0.7310^{\{4\}}$
	α	$0.0055^{\{1\}}$	$0.1827^{\{2\}}$	$0.2398^{\{4\}}$	$0.2807^{\{4\}}$	$-0.0201^{\{2\}}$	$0.1748^{\{1\}}$	$-0.4149^{\{6\}}$	$0.4178^{\{6\}}$	$0.2484^{\{5\}}$	$0.2944^{\{5\}}$	$-0.2319^{\{3\}}$	$0.2399^{\{3\}}$
	β	$0.0044^{\{1\}}$	$0.0480^{\{1\}}$	$1.0646^{\{6\}}$	$1.0753^{\{6\}}$	$-0.1068^{\{2\}}$	$0.1180^{\{2\}}$	$-0.3968^{\{4\}}$	$0.6023^{\{4\}}$	$0.6088^{\{5\}}$	$0.7622^{\{5\}}$	$0.1628^{\{3\}}$	$0.5132^{\{3\}}$
	$\sum$ ranks	7		32		12		25		30		20	
800	b	$0.0412^{\{1\}}$	$0.2749^{\{1\}}$	$-0.5337^{\{4\}}$	$0.5403^{\{4\}}$	$-0.0618^{\{2\}}$	$0.3069^{\{2\}}$	$-0.3443^{\{3\}}$	$0.4824^{\{3\}}$	$-0.5376^{\{5\}}$	$0.5450^{\{5\}}$	$-0.6734^{\{6\}}$	$0.6893^{\{6\}}$
	α	$-0.0008^{\{1\}}$	$0.1645^{\{1\}}$	$0.1447^{\{3\}}$	$0.1867^{\{3\}}$	$0.0098^{\{2\}}$	$0.1655^{\{2\}}$	$-0.3117^{\{6\}}$	$0.3124^{\{6\}}$	$0.2433^{\{5\}}$	$0.2861^{\{5\}}$	$-0.2175^{\{4\}}$	$0.2183^{\{4\}}$
	β	$0.0030^{\{1\}}$	$0.0421^{\{1\}}$	$0.4612^{\{5\}}$	$0.4915^{\{5\}}$	$-0.0995^{\{3\}}$	$0.1007^{\{2\}}$	$-0.1456^{\{4\}}$	$0.4053^{\{4\}}$	$0.5006^{\{6\}}$	$0.6717^{\{6\}}$	$0.0596^{\{2\}}$	$0.2571^{\{3\}}$
	$\sum$ ranks	6		24		13		26		32		25	

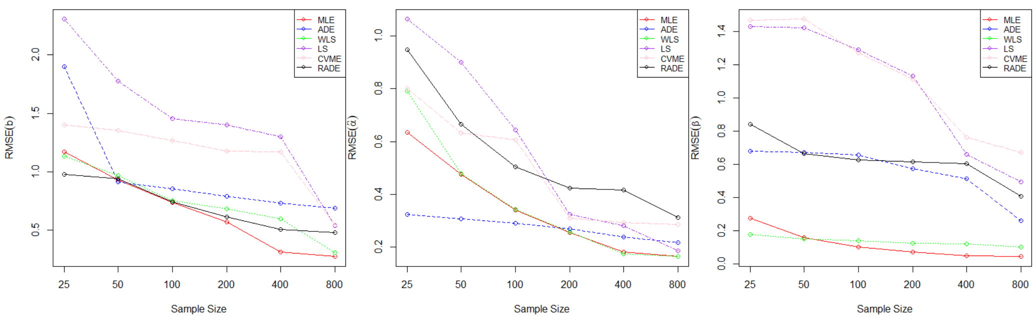

Figure 7: Plots of RMSEs of parameters in Table 2

Table 3: Simulation Results for Different Estimation Methods for $b = 0.7, \alpha = 0.7, \beta = 1.5$

		MLE		LS		WLS		RADE		CVME		ADE	
n	Parameter	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE
25	b	$0.05311^{\{1\}}$	$0.3142^{\{1\}}$	$0.8667^{\{4\}}$	$0.9128^{\{4\}}$	$0.2660^{\{2\}}$	$0.5631^{\{3\}}$	$-0.4083^{\{3\}}$	$0.4823^{\{2\}}$	$0.8915^{\{5\}}$	$0.9450^{\{5\}}$	$1.7888^{\{6\}}$	$1.7953^{\{6\}}$
	α	$0.0537^{\{1\}}$	$0.6125^{\{2\}}$	$-0.4442^{\{3\}}$	$0.4497^{\{1\}}$	$-0.4256^{\{2\}}$	$0.7782^{\{4\}}$	$-1.3216^{\{6\}}$	$1.3440^{\{6\}}$	$-0.8050^{\{5\}}$	$0.8399^{\{5\}}$	$-0.4457^{\{4\}}$	$0.6776^{\{3\}}$
	β	$0.27815^{\{1\}}$	$0.8122^{\{4\}}$	$-0.7365^{\{4\}}$	$0.8473^{\{5\}}$	$-1.2517^{\{6\}}$	$1.3326^{\{6\}}$	$-0.6503^{\{3\}}$	$0.6514^{\{1\}}$	$-0.7620^{\{5\}}$	$0.7699^{\{3\}}$	$-0.6125^{\{2\}}$	$0.7220^{\{2\}}$
	$\sum$ ranks	10		21		23		21		28		23	
50	b	$0.0309^{\{1\}}$	$0.2213^{\{1\}}$	$0.7779^{\{4\}}$	$0.7868^{\{4\}}$	$-0.2368^{\{2\}}$	$0.2502^{\{2\}}$	$-0.4002^{\{3\}}$	$0.4637^{\{3\}}$	$0.8407^{\{5\}}$	$0.8792^{\{5\}}$	$1.7410^{\{6\}}$	$1.7726^{\{6\}}$
	α	$0.03444^{\{1\}}$	$0.4357^{\{2\}}$	$-0.4246^{\{3\}}$	$0.4269^{\{1\}}$	$-0.1443^{\{2\}}$	$0.4823^{\{4\}}$	$-1.0931^{\{6\}}$	$1.0955^{\{6\}}$	$-0.4460^{\{5\}}$	$0.4510^{\{3\}}$	$-0.4299^{\{4\}}$	$0.4837^{\{5\}}$
	β	$0.1236^{\{1\}}$	$0.4319^{\{1\}}$	$-0.7228^{\{4\}}$	$0.7453^{\{4\}}$	$-1.2155^{\{6\}}$	$1.2596^{\{6\}}$	$-0.6366^{\{3\}}$	$0.6371^{\{2\}}$	$-0.7371^{\{5\}}$	$0.7466^{\{5\}}$	$-0.6084^{\{2\}}$	$0.7072^{\{3\}}$
	$\sum$ ranks	7		20		22		23		28		26	
100	b	$0.0181^{\{1\}}$	$0.1511^{\{1\}}$	$0.7655^{\{4\}}$	$0.7675^{\{4\}}$	$-0.2094^{\{3\}}$	$0.2356^{\{3\}}$	$-0.1653^{\{2\}}$	$0.1942^{\{2\}}$	$0.7730^{\{5\}}$	$0.7864^{\{5\}}$	$1.7273^{\{6\}}$	$1.7320^{\{6\}}$
	α	$0.0053^{\{1\}}$	$0.3019^{\{1\}}$	$-0.4225^{\{3\}}$	$0.4248^{\{3\}}$	$0.1066^{\{5\}}$	$0.3097^{\{2\}}$	$-1.0864^{\{6\}}$	$1.0917^{\{6\}}$	$-0.4306^{\{4\}}$	$0.4334^{\{4\}}$	$-0.4178^{\{2\}}$	$0.4728^{\{5\}}$
	β	$0.0502^{\{1\}}$	$0.2508^{\{1\}}$	$-0.6934^{\{4\}}$	$0.6946^{\{4\}}$	$0.8993^{\{6\}}$	$0.9706^{\{6\}}$	$-0.5872^{\{3\}}$	$0.5894^{\{2\}}$	$-0.7203^{\{5\}}$	$0.7376^{\{5\}}$	$-0.5707^{\{2\}}$	$0.6735^{\{3\}}$
	$\sum$ ranks	6		22		25		21		28		24	
200	b	$0.0062^{\{1\}}$	$0.1035^{\{1\}}$	$0.7636^{\{4\}}$	$0.7643^{\{4\}}$	$-0.0989^{\{2\}}$	$0.1667^{\{2\}}$	$-0.1229^{\{3\}}$	$0.1937^{\{3\}}$	$0.7654^{\{5\}}$	$0.7680^{\{5\}}$	$1.4418^{\{6\}}$	$1.4751^{\{6\}}$
	α	$0.0045^{\{1\}}$	$0.2164^{\{1\}}$	$-0.2865^{\{4\}}$	$0.2996^{\{3\}}$	$-0.0868^{\{2\}}$	$0.2577^{\{2\}}$	$-0.1935^{\{3\}}$	$0.9354^{\{6\}}$	$-0.4279^{\{6\}}$	$0.4294^{\{4\}}$	$-0.4061^{\{5\}}$	$0.4673^{\{5\}}$
	β	$0.0225^{\{1\}}$	$0.1660^{\{2\}}$	$-0.4906^{\{3\}}$	$0.5920^{\{5\}}$	$0.0432^{\{2\}}$	$0.1572^{\{1\}}$	$-0.5865^{\{5\}}$	$0.5867^{\{4\}}$	$-0.7091^{\{6\}}$	$0.7141^{\{6\}}$	$-0.5062^{\{4\}}$	$0.5081^{\{3\}}$
	$\sum$ ranks	7		23		11		24		32		29	
400	b	$0.0060^{\{1\}}$	$0.0723^{\{1\}}$	$0.7573^{\{5\}}$	$0.7577^{\{5\}}$	$-0.0500^{\{2\}}$	$0.0747^{\{2\}}$	$-0.1125^{\{3\}}$	$0.1528^{\{3\}}$	$0.7609^{\{6\}}$	$0.7616^{\{6\}}$	$0.6426^{\{4\}}$	$0.6478^{\{4\}}$
	α	$-0.0008^{\{1\}}$	$0.1487^{\{2\}}$	$-0.2798^{\{4\}}$	$0.2859^{\{3\}}$	$-0.0245^{\{2\}}$	$0.1461^{\{1\}}$	$-0.1914^{\{3\}}$	$0.9149^{\{6\}}$	$-0.4247^{\{6\}}$	$0.4261^{\{4\}}$	$-0.3869^{\{5\}}$	$0.4298^{\{5\}}$
	β	$0.0144^{\{1\}}$	$0.1171^{\{1\}}$	$-0.2927^{\{3\}}$	$0.3495^{\{3\}}$	$0.0388^{\{2\}}$	$0.1337^{\{2\}}$	$-0.5618^{\{5\}}$	$0.5634^{\{5\}}$	$-0.5646^{\{6\}}$	$0.6927^{\{6\}}$	$-0.4011^{\{4\}}$	$0.4628^{\{4\}}$
	$\sum$ ranks	7		23		11		25		34		26	
800	b	$0.0040^{\{1\}}$	$0.0635^{\{2\}}$	$0.5390^{\{6\}}$	$0.5413^{\{6\}}$	$-0.0258^{\{2\}}$	$0.0544^{\{1\}}$	$0.0992^{\{3\}}$	$0.1096^{\{3\}}$	$0.4578^{\{5\}}$	$0.5217^{\{5\}}$	$0.3930^{\{4\}}$	$0.4210^{\{4\}}$
	α	$-0.0003^{\{1\}}$	$0.1353^{\{1\}}$	$-0.2182^{\{4\}}$	$0.2580^{\{3\}}$	$-0.0223^{\{2\}}$	$0.1459^{\{2\}}$	$-0.1781^{\{3\}}$	$0.2781^{\{4\}}$	$-0.2769^{\{6\}}$	$0.2832^{\{5\}}$	$-0.2634^{\{5\}}$	$0.2979^{\{6\}}$
	β	$0.0037^{\{1\}}$	$0.0806^{\{1\}}$	$-0.1966^{\{3\}}$	$0.1970^{\{4\}}$	$0.0306^{\{2\}}$	$0.0938^{\{2\}}$	$-0.3083^{\{6\}}$	$0.5083^{\{6\}}$	$-0.3044^{\{5\}}$	$0.3757^{\{5\}}$	$-0.2631^{\{4\}}$	$0.1687^{\{3\}}$
	$\sum$ ranks	7		26		11		25		31		26	

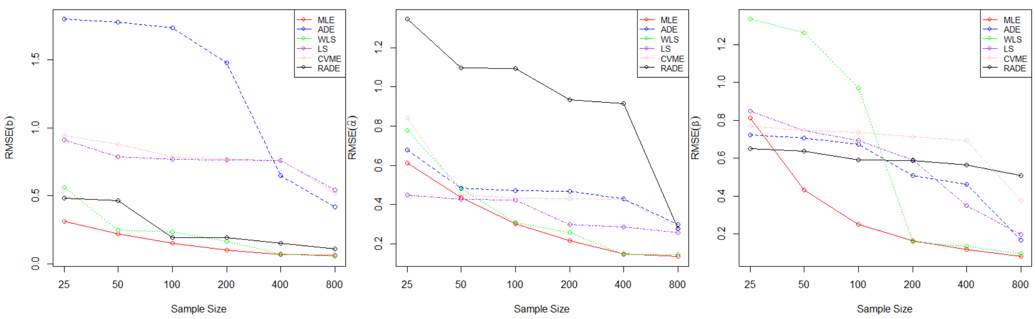

Figure 8: Plots of RMSEs of parameters in Table 3

Table 4: Partial and Overall Ranks of all Estimation Methods of TL-Exp-W Distribution by Various Model Parameter Values

Parameters	n	MLE	LS	WLS	RADE	CVME	ADE
	25	1	6	2	4	5	3
	50	1	6	3	4	5	2
$b = 2.0, \alpha = 0.5, \beta = 0.5.$	100	1	6	2	3.5	5	3.5
	200	1	6	2	3.5	5	3.5
	400	1	6	2	4	5	3
	800	1	3	2	5	6	4
	25	1	2.5	4.5	2.5	6	4.5
	50	1	2	3	4	6	5
$b = 0.7, \alpha = 0.7, \beta = 1.5.$	100	1	3	5	2	6	4
	200	1	3	2	4	6	5
	400	1	3	2	4	6	5
	800	1	4.5	2	3	6	4.5
$\sum$ ranks		12	51	31.5	43.5	67	47
Overall rank		1	5	2	3	6	4

In Tables 2 and 3, the row indicating $\sum$ Ranks represents the partial sum of the ranks. Among all the estimators for a given metric, the superscript indicates their rank. Table 2 presents, for example, the ABIAS of $\hat{b}$ obtained via MLE method as $0.2683^{\{1\}}$ for $n = 25$. This indicates that the ABIAS of $\hat{b}$ obtained using the MLE method ranks first among all other estimators.

Table 4 shows the partial and overall ranks of all the estimation methods of TL-Exp-W distribution by various model parameter values. Based on the results in Tables 2 and 3, the TL-Exp-W distribution is stable, as the ABIAS and RMSE values for its three parameters are modest. With increasing sample size, the bias occasionally decreases while the RMSE decreases for all estimations as the sample size increases. In general, all estimation methods provide accurate bias and mean squared error estimates for large sample sizes. Table 4 shows that MLE method allows us to obtain better estimates of TL-Exp-W parameters, followed by WLS and then RADE methods. According to the rankings, the CVME method performs the least well.

8 Applications

In this section, we illustrate the performance of the TL-Exp-G family of distributions by fitting the special case of the TL-Exp-W distribution to two real data sets. The performance of the models were assessed using the well recognized goodness-of-fit statis-

tics, namely, -2loglikelihood (-2 log (L)), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), Cramer-Von Mises (W^*) and Andersen-Darling (A^*) as described by Chen and Balakrishnan (1995). We also computed the Kolmogorov-Smirnov (K-S) goodness-of-fit statistic. The model with the smallest values of these goodness-of-fit statistics and a higher p-value for the K-S statistic, is regarded as the best fitting model.
The package *AdequacyModel* in R software for goodness-of-fit statistics was used in the analysis. Model parameter estimates (standard errors in parenthesis) and the goodness-of-fit-statistics for the two data sets are shown in Tables 5 and 6. We also present plots of the fitted densities, the histogram of the data and probability plots (Chambers et al. (1983)) to show how well our model fits the observed data sets.
We compared the TL-Exp-W distribution with several other non-nested models, namely Topp-Leone exponential exponential (TLE-E) by Sanusi et al. (2020), Topp-Leone inverse Gompertz (TLIG) by Adegoke et al. (2023), Type I Half Logistic-Weibull (TIHL-W) by Kumar et al. (2015), Topp-Leone-Weibull (TL-W) by Rezaei et al. (2017), Weibull-Lomax (WLx) by Jamal et al. (2019) and exponentiated half logistic (EHL) by Johnson et al. (1994). The pdfs of these distributions are given in the appendix.

We present plots of fitted densities, histograms of the data and probability plots for each example to show how well our model fits the observed data sets. To obtain the probability plot, we plotted $F_{TL-Exp-W}(x_{(j)}; \hat{b}, \hat{\alpha}, \hat{\beta})$ against $\dfrac{j-0.375}{n+0.25}$, $j = 1, 2, \cdots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measures of closeness are given by the sum of squares $SS = \sum_{j=1}^{n}\left[F_{TL-Exp-W}(x_{(j)}; \hat{b}, \hat{\alpha}, \hat{\beta}) - \left(\dfrac{j-0.375}{n+0.25}\right)\right]^2$.

In addition, profile log-likelihood plots, empirical cumulative distribution function (ECDF), Kaplan-Meier (K-M) survival curve, total time on test (TTT) plots and hrf plots are presented.

8.1 Environmental Data

The first data measures the acidity of rainfalls for forty days in the state of Minnesota. This data set was analyzed by Elbatal et al. (2022). (**See the data in the Appendix**).

The estimated variance-covariance matrix for TL-Exp-W model on the environmental data set is given by

$$\begin{pmatrix} 9.7423\times10^{-04} & 2.0471\times10^{-07} & 1.0242\times10^{-03} \\ 2.0471\times10^{-07} & 5.6534\times10^{-11} & 1.0817\times10^{-07} \\ 1.0242\times10^{-03} & 1.0817\times10^{-07} & 1.9249\times10^{-03} \end{pmatrix},$$

and the 95% confidence intervals for the model parameters are given by
$b \in [0.1310 \pm 6.1177\times10^{-02}]$, $\alpha \in [344.4800 \pm 1.4737\times10^{-05}]$ and $\beta \in [1.1078 \pm 8.5993\times10^{-02}]$, respectively.

Table 5: Parameter estimates and goodness-of-fit statistics for various models fitted to the environmental data

	Estimates			Statistics							
Model	b	α	β	$-2\log(L)$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
TL-Exp-W	0.1310	344.4800	1.1078	93.06351	99.06348	99.73015	104.1301	0.0361	0.2793	0.0727	0.9839
	(3.1213×10^{-02})	(7.5190×10^{-06})	(4.3874×10^{-02})								
	σ	λ	γ								
TLE-E	86.7890	20.0330	0.0326	94.01098	100.0109	100.6776	105.0776	0.0404	0.3152	0.0767	0.9725
	(9.1007×10^{-08})	(1.7611×10^{-06})	(1.1431×10^{-03})								
	α	λ	β								
TLIG	0.3245	0.8847	12.5096	98.59627	104.5963	105.2629	109.6629	0.0921	0.6551	0.1129	0.6873
	(0.4684)	(0.6263)	(2.8536)								
	a	b	α								
TLW	0.1949	40.9058	0.0683	104.0022	110.0021	110.6688	115.0688	0.2755	1.7571	0.1710	0.1925
	(0.0026)	(4.5449)	(0.0136)								
	λ	δ	γ								
TIHLW	0.0319	0.1514	4.1700	97.87511	103.8751	104.5418	108.9417	0.1057	0.6958	0.1048	0.7710
	(0.0246)	(0.0052)	(0.5380)								
	a	λ									
EHL	13.9756	1.0097	-	102.3704	106.3704	106.6947	109.7482	0.0396	0.3052	0.1538	0.3002
	(6.2805)	(0.1357)									
WLx	0.2984	2.3457	9.1626	95.59287	101.5929	102.2595	106.6595	0.0773	0.5248	0.0967	0.8478
	(0.1090)	(2.2077)	(2.7330)								

Table 5 indicates that TL-Exp-W has the lowest values of all goodness-of-fit statistics and the highest p-value for the K-S statistic. Therefore, we can conclude that the TL-Exp-W model performs better than the other models presented in Table 5 for the environmental data. In addition, Figure 9 shows that the TL-Exp-W outperforms the competing non-nested models on the environmental data. Figure 10 shows the profile log-likelihood plots for parameters of the TL-Exp-W distribution for the environmental data. It can be seen that the MLEs of the TL-Exp-W distribution can be uniquely identified.

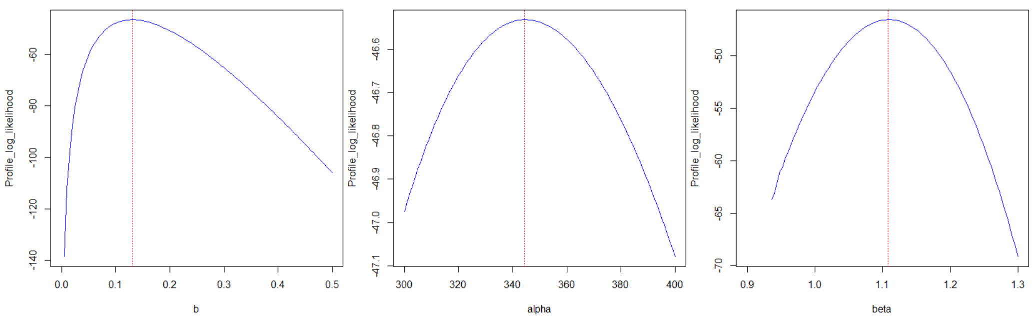

Figure 10: Profile log-likelihood function plots for parameters of TL-Exp-W distribution on the environmental data

Figure 11 shows the cdf curve and the survival function of the TL-Exp-W distribution

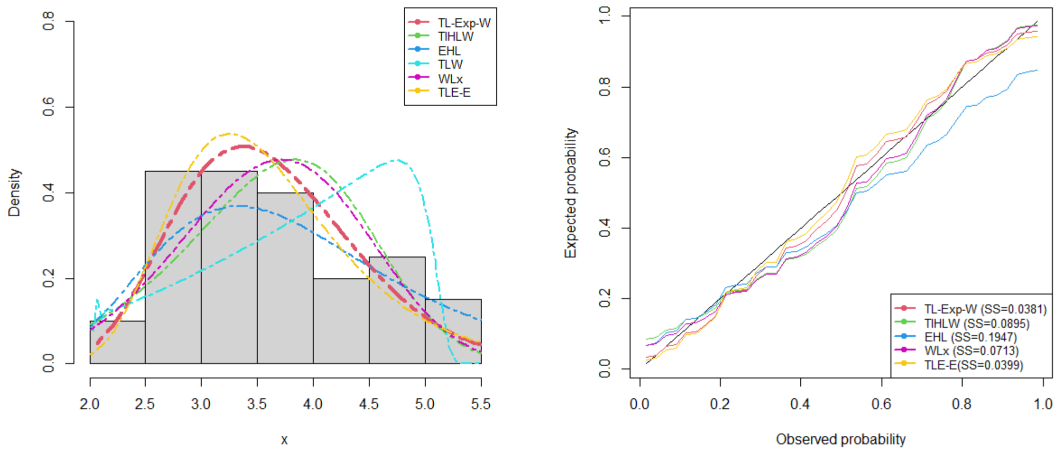

Figure 9: Fitted density and probability plots for the environmental data

in green superimposed on the empirical cdf and Kaplan-Meier (K-M) curve in black. It can be seen that the plots are close to each other which indicate that our model is explaining the characteristics of the environmental data. The TTT plot for the environmental data indicates an increasing hazard rate function and our model correctly captured the hazard rate function as shown in Figure 12.

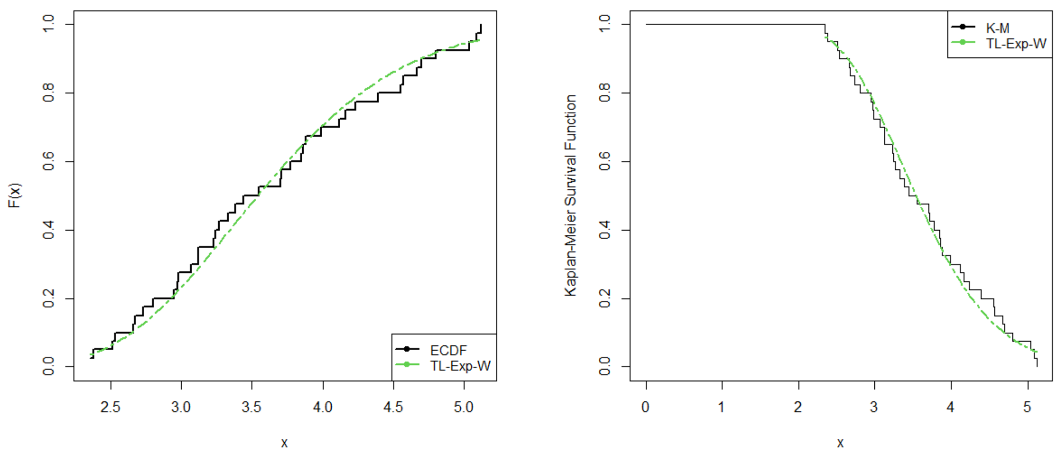

Figure 11: Estimated cdf and Kaplan-Meier survival plot for the environmental data

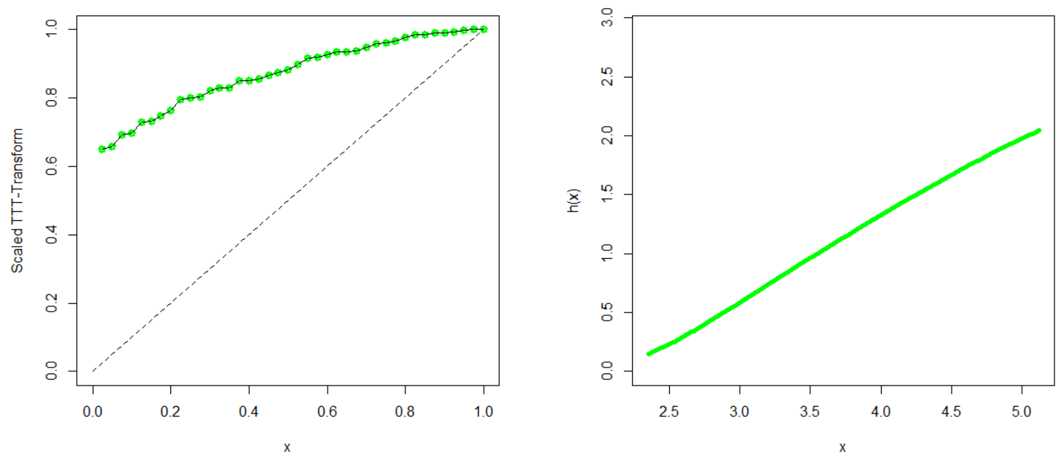

Figure 12: Scaled TTT-Transform plots and fitted hazard rate function for the TL-Exp-W distribution for environmental data

8.2 Remission Times Data

The second data set is on the remission times (months) of 128 bladder cancer patients (Lee et al. (2003)). (**See the data in the Appendix**).

The estimated variance-covariance matrix for the TL-Exp-W model on the remission times data is given by

$$\begin{pmatrix} 0.3919 & -0.5928 & -0.0035 \\ -0.5928 & 1.0158 & 0.0081 \\ -0.0035 & 0.0081 & 0.0003 \end{pmatrix},$$

and the 95% confidence intervals for the model parameters are given by
$b \in [2.2447 \pm 1.2269]$, $\alpha \in [3.8082 \pm 1.9754]$ and $\beta \in [0.3668 \pm 0.0324]$, respectively.

Table 6: Parameter estimates and goodness-of-fit statistics for various models fitted to the remission times data

Model	Estimates			Statistics							
Model	b	α	β	$-2\log(L)$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
TL-Exp-W	2.2447	3.8082	0.3668	821.1111	827.1112	827.3047	835.6672	0.0317	0.22258	0.0537	0.8537
	(0.6260)	(1.0078)	(0.0165)								
	σ	λ	γ								
TLE-E	1.2169	872.0400	6.9270×10^{-05}	826.1947	832.1947	832.3883	840.7508	0.1125	0.6764	0.0733	0.497
	(2.0209×10^{-10})	(4.4460×10^{-13})	(5.6105×10^{-06})								
	α	λ	β								
TLIG	1.7834	2.5317	5.6548×10^{-09}	1032.851	1038.86	1039.0530	1047.416	1.0595	6.2999	0.3439	< 0.0001
	(0.0723)	(0.0788)	(1.5866×10^{-04})								
	a	b	α								
TLW	0.1735	0.5000	5.0413	822.9283	828.9283	829.1218	837.4844	0.0600	0.4113	0.5394	< 0.0001
	(0.0868)	(0.1033)	(2.5363)								
	λ	δ	γ								
TIHLW	0.4391	0.4588	0.8880	830.1918	836.1918	836.3854	844.7479	0.1660	0.9780	0.0765	0.4420
	(0.0412)	(0.0395)	(0.0603)								
	a	λ									
EHL	0.9527	0.14397	-	833.2714	837.2714	837.3674	842.9754	0.2171	1.2739	0.0949	0.1994
	(0.1091)	(0.0144)									
WLx	0.1000	112.4400	5.1888	823.6775	829.6775	829.8711	838.2336	0.0737	0.4682	0.0581	0.7795
	(0.0012)	(0.0024)	(0.3527)								

Table 6 presents the goodness-of-fit statistics for various models fitted to the remission times data. The TL-Exp-W distribution has the highest p-value for the K-S statistic and the lowest values for all goodness-of-fit statistics. Thus, we conclude that the TL-Exp-W model performs better on the remission times data than the non-nested presented in Table 6. Figure 13 shows the histogram, fitted densities and probability plots on the remission times data. It can be seen that TL-Exp-W outperforms the competing models in capturing the characteristics of the remission times data. Figure 14 shows the profile log-likelihood plots for parameters of the TL-Exp-W distribution on the remission times data. This shows that the MLEs of the TL-Exp-W distribution are identifiable.

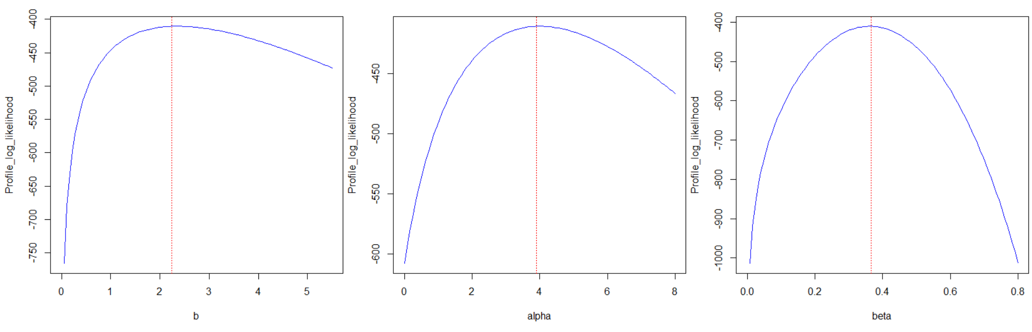

Figure 14: Profile log-likelihood function plots for parameters of TL-Exp-W distribution on the remission times data

In Figure 15, we see that the fitted cdf for the TL-Exp-W distribution is closer to the empirical cdf while the survival function in blue is also close to the Kaplan-Meier (K-M)

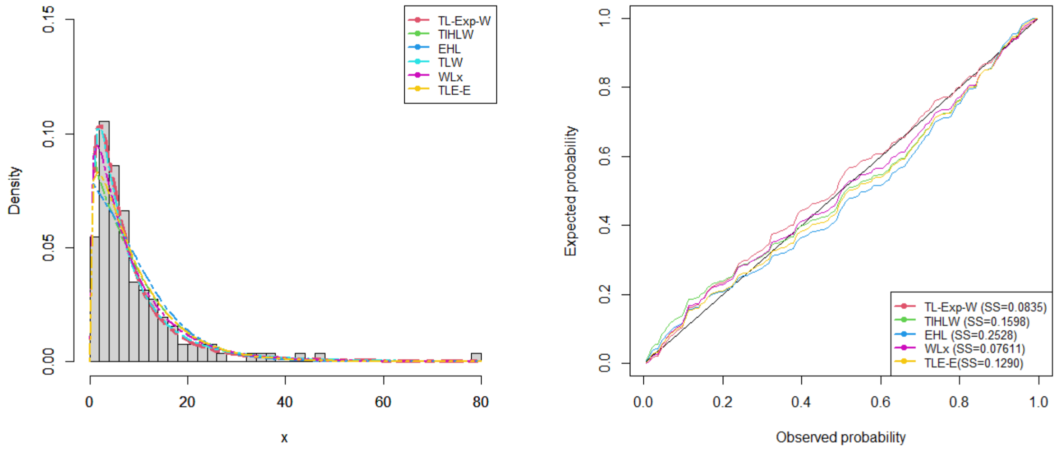

Figure 13: Fitted density and probability plots for the remission times data

curve which indicate that our model is working properly in explaining the remission times data. The TTT plot for the remission times data indicates an upside-down hazard rate function and it is correctly captured by our model as shown in Figure 16.

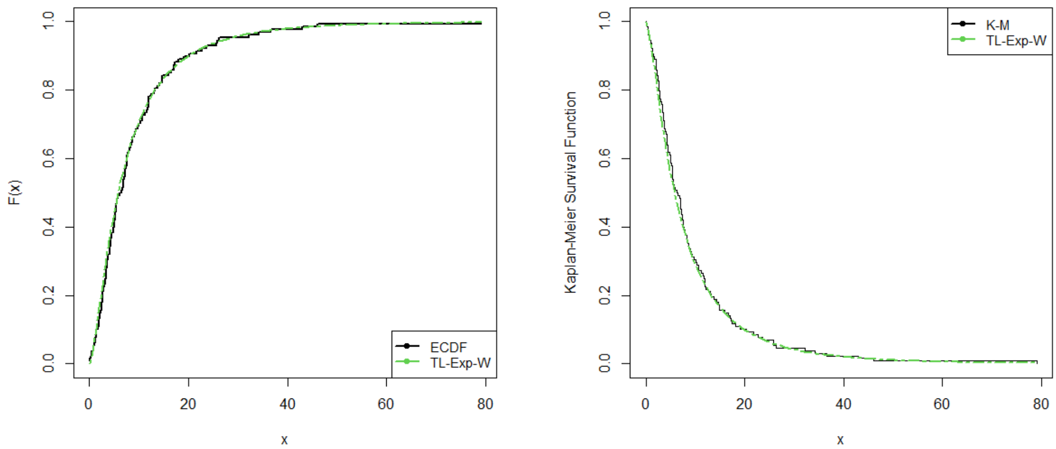

Figure 15: Estimated cdf and Kaplan-Meier survival plot for the remission times data

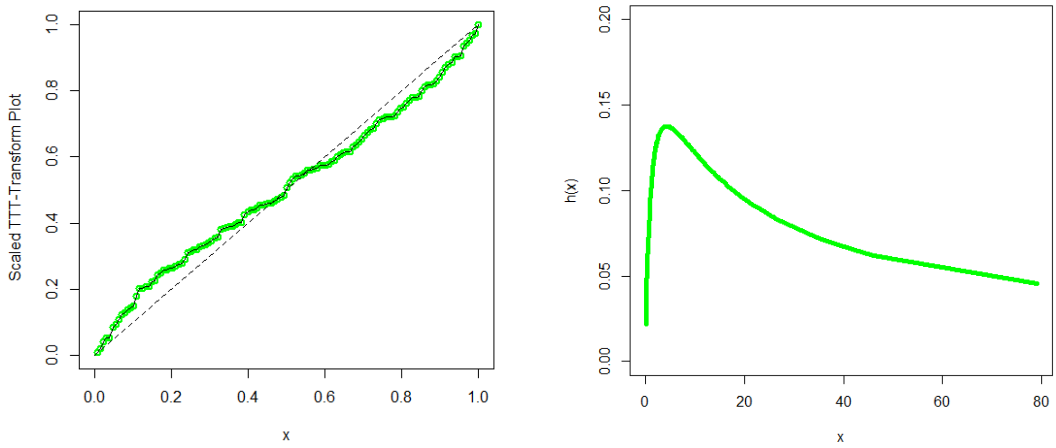

Figure 16: Scaled TTT-Transform plots and fitted hazard rate function for the TL-Exp-W distribution for remission times data

9 Censoring

In this section, we estimate the parameters of the TL-Exp-W distribution and apply the results in a scenario with type I right censoring.

9.1 Type I Right Censoring

This is the most common form of incomplete data often encountered in survival analysis. Type I censoring arises when the study is conducted over a specified time period that can terminate before all the units have failed. Consider a sample of size n of independent positive random variables $X_1, ..., X_n$, such that X_i is associated with an indicator variable $\epsilon_i = 0$ if X_i is a censoring time. Let $\Delta = (b, \alpha, \varphi)^T$ be the vector of model parameters, then the log-likelihood function $\ell_n = \ell_n(\Delta)$ of a type I right censored sample based on a random sample of size n from the TL-Exp-G family of distributions is given by

$$
\begin{aligned}
\ell_n(\Delta) &= \sum_{i=1}^{n} \epsilon_i \ln(2b) - \alpha \sum_{i=1}^{n} \epsilon_i \bar{G}(x_i;\varphi) + \sum_{i=1}^{n} \epsilon_i \ln(1+\alpha G(x_i;\varphi)) \\
&+ \sum_{i=1}^{n} \epsilon_i \ln\left(1 - G(x_i;\varphi) e^{-\alpha \bar{G}(x_i;\varphi)}\right) + \sum_{i=1}^{n} \epsilon_i \ln(g(x_i;\varphi)) \\
&+ (b-1) \sum_{i=1}^{n} \epsilon_i \ln\left(1 - \left(1 - G(x_i;\varphi) e^{-\alpha \bar{G}(x_i;\varphi)}\right)^2\right) \\
&+ \sum_{i=1}^{n} (1-\epsilon_i) \ln\left[1 - \left[1 - \left(1 - G(x_i;\varphi) e^{-\alpha \bar{G}(x_i;\varphi)}\right)^2\right]^b\right].
\end{aligned}
$$

In order to obtain the estimates of the unknown parameters from the TL-Exp-G family of distributions under type I right censoring, we solve $U = (\frac{\partial \ell_n}{\partial b}, \frac{\partial \ell_n}{\partial \alpha}, \frac{\partial \ell_n}{\partial \varphi_k})^T = \mathbf{0}$, using a numerical method such as Newton-Raphson procedure. Elements of the score vector are given in the appendix.

9.2 Censoring Application: Maintenance Data

The set of data is the maintenance data with 46 observations reported on active repair times (hours) for an airborne communication transceiver. The data given has eight observations removed to illustrate type I right censoring. The data was analyzed by Fagbamigbe et al. (2019). (**See the data in the Appendix**).

The estimated variance-covariance matrix for the TL-Exp-W model on the type I right censored maintenance data is given by

$$
\begin{pmatrix}
0.0015 & -0.6199 & -0.0003 \\
-0.6199 & 298.7189 & 0.1796 \\
-0.0003 & 0.1796 & 0.0017
\end{pmatrix},
$$

and the 95% confidence intervals for the model parameters are given by
$b \in [0.0883 \pm 0.0748]$, $\alpha \in [40.4761 \pm 33.8756]$ and $\beta \in [0.4820 \pm 0.0808]$, respectively.

Table 7: Parameter estimates and goodness-of-fit statistics for various models fitted to the type I right censored maintenance data

	Estimates			Statistics			
Model	b	α	β	$-2\log(L)$	AIC	$AICC$	BIC
TL-Exp-W	0.0883	40.4761	0.4820	190.3708	196.3708	196.9423	201.8568
	(0.0382)	(17.2834)	(0.0412)				
	α	λ	β				
TLIG	1.4968	1.1061	7.7514×10^{-08}	191.7579	197.7572	198.3286	203.2431
	(0.4321)	(0.2263)	(0.0194)				
	a	b	α				
TLW	0.0437	0.9345	0.5830	212.5334	218.5334	219.1048	224.0193
	(0.0167)	(0.4141)	(0.3856)				
	λ	δ	γ				
TIHLW	4.2694×10^{04}	2.0008×10^{-05}	0.4496	194.8727	200.8729	201.4443	206.3588
	(1.2264×10^{-15})	(2.6169×10^{-06})	(7.3415×10^{-11})				
	a	λ					
EHL	0.6251	0.2251	-	196.2472	200.2472	200.5262	203.9045
	(0.1209)	(0.0538)					
WLx	89.1000	1.1029	0.0048	271.2388	277.2388	277.8103	282.7248
	(6.5935×10^{-08})	(0.5072)	(0.0010)				

From the results given in Table 7, it is clear that the TL-Exp-W distribution outperforms the other fitted distributions since it has the lowest values of the goodness-of-fit statistics: $-2\ln(L), AIC, CAIC$, and BIC.

10 Concluding Remarks

In this paper, we developed and presented a new extended family of generalized distributions called the Topp-Leone-Exponential generator (TL-Exp-G) of distributions. Several statistical properties of the new family of distributions are presented. Different estimation methods are considered to estimate the parameters of the TL-Exp-G family of distributions. Monte Carlo simulations were used to evaluate the consistency property of the estimation methods for a special case of the TL-Exp-G distribution. Lastly, we illustrate the usefulness of the new family of distributions by fitting a special case of the TL-Exp-G family to complete and censored data sets. In the future, we will seek to estimate the parameters of the new family of distributions using Bayesian technique. The new family of distributions can also be generalized using bivariate and multivariate extensions.

Appendix

Click on the link below for results in the appendix.
`https://drive.google.com/file/d/1ZstUoRRdHttF8CTegZK6DEyKQLeRbCuy/view?usp=sharing`

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