Mathematical Appendix to "Interconvertible Rules"

M1: Proof of Rule 1

$$\Omega \log\left(\frac{\Sigma p_{1i}x_i}{\Sigma p_{0i}x_i}\right) \Rightarrow \Omega \sum \left(\frac{L(p_{1i}x_i, p_{0i}x_i)}{L(\Sigma p_{1i}x_i, \Sigma p_{0i}x_i)}\right) \log\left(\frac{p_{1i}x_i}{p_{0i}x_i}\right)$$

$$= \Omega \sum \left(\frac{p_{1i}x_i - p_{0i}x_i}{\log((p_{1i}x_i)/(p_{0i}x_i))}\right)\left(\frac{\log((\Sigma p_{1i}x_i)/(\Sigma p_{0i}x_i))}{\Sigma p_{1i}x_i - \Sigma p_{0i}x_i}\right) \log\left(\frac{p_{1i}x_i}{p_{0i}x_i}\right)$$

$$= \Omega \log \frac{\Sigma p_{1i}x_i}{\Sigma p_{0i}x_i}.$$

M2: Proof of Rule 2

Since $\Sigma L(v_{1i}, v_{0i}) \log(v_{1i}/v_{0i}) = 0$, we obtain

$$\sum L(v_{1i}, v_{0i}) \log \frac{(p_{1i}x_i)/(\Sigma p_{1i}x_i)}{(p_{0i}x_i)/(\Sigma p_{0i}x_i)} = \sum L(v_{1i}, v_{0i}) \left(\log \frac{p_{1i}x_i}{p_{0i}x_i} - \log \frac{\Sigma p_{1i}x_i}{\Sigma p_{0i}x_i}\right) = 0.$$

$$\therefore \log \frac{\Sigma p_{1i}x_i}{\Sigma p_{0i}x_i} = \sum \frac{L(v_{1i}, v_{0i})}{\Sigma L(v_{1i}, v_{0i})} \log\left(\frac{p_{1i}x_i}{p_{0i}x_i}\right).$$

M3: Proof of Rule 3

$$\sum \gamma_i \log\left(\frac{p_{1i}}{p_{0i}}\right) \Rightarrow L(\Sigma p_{1i}x_i, \Sigma p_{0i}x_i) \log\left(\frac{\Sigma p_{1i}x_i}{\Sigma p_{0i}x_i}\right) = \Sigma(p_{1i}x_i - p_{0i}x_i)$$

$$= \sum L(p_{1i}x_i, p_{0i}x_i) \log\left(\frac{p_{1i}x_i}{p_{0i}x_i}\right) = \sum x_i L(p_{1i}, p_{0i}) \log\left(\frac{p_{1i}}{p_{0i}}\right).$$

M4: Proof of Rule 4

From (6), we have

$$\log\left(\frac{\Sigma p_{1i}z_i}{\Sigma p_{0i}z_i}\right) = \sum \left(\frac{L(v_{1i}, v_{0i})}{\Sigma L(v_{1i}, v_{0i})}\right) \log\left(\frac{p_{1i}}{p_{0i}}\right)$$

where

$v_{1i} = (p_{1i}z_i)/(\Sigma p_{1i}z_i)$, $v_{0i} = (p_{0i}z_i)/(\Sigma p_{0i}z_i)$, and $L(v_{1i}, v_{0i})/\big(\Sigma L(v_{1i}, v_{0i})\big) = \delta_i$.
Using $v_{1i}/v_{0i} = (p_{1i}\Sigma p_{0i}z_i)/(p_{0i}\Sigma p_{1i}z_i) = p_{1i}/(p_{0i}B)$, we have

$$L(v_{1i}, v_{0i}) = v_{0i}L(v_{1i}/v_{0i}, 1) = v_{0i}L(p_{1i}/(p_{0i}B), 1) = (v_{0i}/(p_{0i}B))L(p_{1i}, p_{0i}B).$$

Thus $\delta_i = (v_{0i}/p_{0i})L(p_{1i}, p_{0i}B)/\big(\Sigma(v_{0i}/p_{0i})L(p_{1i}, p_{0i}B)\big)$.

$$\therefore v_{0i} = \Big(p_{0i}\delta_i\big(\Sigma(v_{0i}/p_{0i})L(p_{1i}, p_{0i}B)\big)\Big)/\big(L(p_{1i}, p_{0i}B)\big). \qquad (1)$$

Since $\Sigma v_{0i} = 1$, we have

$$\Sigma(v_{0i}/p_{0i})L(p_{1i},p_{0i}B) = 1/\big(\Sigma(p_{0i}\delta_i/L(p_{1i},p_{0i}B))\big). \qquad (2)$$

From (1) and (2), we derive

$$v_{0i} = (p_{0i}z_i)/(\Sigma p_{0i}z_i) = (p_{0i}\delta_i/L(p_{1i},p_{0i}B))/(\Sigma p_{0i}\delta_i/L(p_{1i},p_{0i}B)).$$

Hence

$$z_i = C\,\delta_i/L(p_{1i},p_{0i}B)$$

where *C* is any predetermined value. We set *C* = 1 for simplicity.

M5: Proofs of (8) and (9)

M5-1 Rule 2⊗Rule 3

This combined rule yields:

$$\Omega\log\left(\frac{\Sigma p_{1i}x_i}{\Sigma p_{0i}x_i}\right) \Rightarrow \Omega\Sigma\beta_i\log\left(\frac{p_{1i}}{p_{0i}}\right) \Rightarrow \Omega\Psi\log\left(\frac{\Sigma p_{1i}z_i}{\Sigma p_{0i}z_i}\right)$$

where $\Sigma\beta_i = 1$ and $\Psi \geq \Sigma\beta_i = 1$. Thus $\Omega \leq \Omega\Psi$, which implies (8).

M5-2 Rule 1⊗Rule 4

This yields:

$$\Omega\log\left(\frac{\Sigma p_{1i}x_i}{\Sigma p_{0i}x_i}\right) \Rightarrow \Omega\Sigma\alpha_i\log\left(\frac{p_{1i}}{p_{0i}}\right) \Rightarrow \Omega\,(\Sigma\alpha_i)\log\left(\frac{\Sigma p_{1i}z_i}{\Sigma p_{0i}z_i}\right)$$

where $\Sigma\alpha_i \leq 1$. Thus $\Omega \geq \Omega\Sigma\alpha_i$, which implies (9).

M6: Proofs of (14) and (15)

M6-1 Rule 3⊗Rule 2

This combined rule is

$$\sum\gamma_i\log\left(\frac{p_{1i}}{p_{0i}}\right) \Rightarrow \Omega\log\left(\frac{\Sigma p_{1i}x_i}{\Sigma p_{0i}x_i}\right) \Rightarrow \Omega\sum\beta_i\log\left(\frac{p_{1i}}{p_{0i}}\right)$$

where $\Omega \geq \Sigma\gamma_i \; and \; \Sigma\beta_i = 1$. Thus, we have (14).

M6-2 Rule 4⊗Rule 1

This combined rule derives

$$\sum\gamma_i\log\left(\frac{p_{1i}}{p_{0i}}\right) \Rightarrow (\Sigma\gamma_i)\log\left(\frac{\Sigma p_{1i}z_i}{\Sigma p_{0i}z_i}\right) \Rightarrow (\Sigma\gamma_i)\sum\alpha_i\log\left(\frac{p_{1i}}{p_{0i}}\right)$$

where $\Sigma\alpha_i \leq 1$. Thus, we have (15).

M7: Proof of (27)

$$\partial U_t/\partial q_{ti} = (-1/\gamma)\,(\Sigma a_i(q_{ti})^{-\gamma})^{(1+\gamma)/(-\gamma)}(-\gamma a_i(q_{ti})^{-\gamma-1})$$
$$= (U_t)^{1+\gamma}a_i(q_{ti})^{-\gamma-1} = \lambda_t p_{ti}.$$

$$\therefore\; (U_t)^{1+\gamma}a_i(q_{ti})^{-\gamma} = \lambda_t p_{ti}q_{ti}. \qquad (1)$$

This yields

$(U_t)^{1+\gamma}\Sigma a_i(q_{ti})^{-\gamma} = \lambda_t \Sigma p_{ti}q_{ti} = \lambda_t y_{tt}$. (2)

Combining (1) and (2) leads to

$(a_i(q_{ti})^{-\gamma})/(\Sigma a_i(q_{ti})^{-\gamma}) = (p_{ti}q_{ti})/y_{tt} = w_{tti} = v_{ti}$.

Hence $L(v_{1i}, v_{0i}) = L(w_{11i}, w_{00i})$.

M8: Proof of (31)

Since $z_i = a_i/L(c_{1i}, c_{0i}B) \approx a_i/(G(c_{1i}, c_{0i})B^{0.5})$ and

$a_i = \left(G(p_{1i}, p_{0i})G(c_{1i}, c_{0i})\right)/G(m_{11}, m_{00})$,

we have

$$\frac{\Sigma c_{1i}z_i}{\Sigma c_{0i}z_i} \approx \frac{\Sigma\left(c_{1i}G(p_{1i},p_{0i})/\left(G(m_{11},m_{00})B^{0.5}\right)\right)}{\Sigma(c_{0i}G(p_{1i},p_{0i})/(G(m_{11},m_{00})B^{0.5}))} = \frac{\Sigma c_{1i}G(p_{1i},p_{0i})}{\Sigma c_{0i}G(p_{1i},p_{0i})}.$$

M9: Proof of Rule 1⊗Rule 3 (Appendix A.1)

$$\Omega \log\left(\frac{\Sigma p_{1i}x_i}{\Sigma p_{0i}x_i}\right) \Rightarrow \Omega\Sigma\alpha_i \log\left(\frac{p_{1i}}{p_{0i}}\right) \Rightarrow \Omega\Psi \log\left(\frac{\Sigma p_{1i}z_i}{\Sigma p_{0i}z_i}\right).$$

where

$$\alpha_i = \frac{L(p_{1i}x_i, p_{0i}x_i)}{L(\Sigma p_{1i}x_i, \Sigma p_{0i}x_i)}, z_i = \frac{\alpha_i}{L(p_{1i}, p_{0i})} = \frac{x_i}{L(\Sigma p_{1i}x_i, \Sigma p_{0i}x_i)},$$

and

$$\Psi = L(\Sigma p_{1i}z_i, \Sigma p_{0i}z_i) = \frac{L(\Sigma p_{1i}x_i, \Sigma p_{0i}x_i)}{L(\Sigma p_{1i}x_i, \Sigma p_{0i}x_i)} = 1.$$

M10: Proof of Rule 2⊗Rule 4 (Appendix A.2)

$$\Omega \log\left(\frac{\Sigma p_{1i}x_i}{\Sigma p_{0i}x_i}\right) \Rightarrow \Omega\Sigma\beta_i \log\left(\frac{p_{1i}}{p_{0i}}\right) \Rightarrow \Omega(\Sigma\beta_i) \log\left(\frac{\Sigma p_{1i}z_i}{\Sigma p_{0i}z_i}\right)$$

where

$$\beta_i = \frac{L(v_{1i}, v_{0i})}{\Sigma L(v_{1i}, v_{0i})}, \Sigma\beta_i = 1, v_{1i} = \frac{p_{1i}x_i}{\Sigma p_{1i}x_i}, v_{0i} = \frac{p_{0i}x_i}{\Sigma p_{0i}x_i}, z_i = \frac{\delta_i}{L(p_{1i}, p_{0i}B)}, \delta_i = \beta_i,$$

and

$$B = \frac{\Sigma p_{1i}x_i}{\Sigma p_{0i}x_i} \left(= \frac{\Sigma p_{1i}z_i}{\Sigma p_{0i}z_i}\right).$$

Here, $\frac{v_{1i}}{v_{0i}} = \left(\frac{p_{1i}}{p_{0i}}\right)\left(\frac{\Sigma p_{0i}x_i}{\Sigma p_{1i}x_i}\right) = \frac{p_{1i}}{p_{0i}B}$. Thus

$$L(v_{1i}, v_{0i}) = v_{0i}L(v_{1i}/v_{0i}, 1) = (v_{0i}/(p_{0i}B))L(p_{1i}, p_{0i}B)$$
$$= (x_i/(\Sigma p_{1i}x_i))L(p_{1i}, p_{0i}B).$$

$$\therefore\ z_i = \frac{x_i/(\Sigma p_{1i}x_i)}{\Sigma(x_i/(\Sigma p_{1i}x_i))L(p_{1i},p_{0i}B)} = \frac{x_i}{\Sigma x_i L(p_{1i},p_{0i}B)}.$$

M11: Proof of Rule 3⊗Rule 1 (Appendix A.3)

$$\sum \alpha_i \log\left(\frac{p_{1i}}{p_{0i}}\right) \Rightarrow \Omega \log\left(\frac{\Sigma p_{1i}x_i}{\Sigma p_{0i}x_i}\right) \Rightarrow \Omega \sum \beta_i \log\left(\frac{p_{1i}}{p_{0i}}\right).$$

where $x_i = \alpha_i/L(p_{1i},p_{0i})$, $\Omega = L(\Sigma p_{1i}x_i, \Sigma p_{0i}x_i)$, and

$$\beta_i = \frac{L(p_{1i}x_i, p_{0i}x_i)}{L(\Sigma p_{1i}x_i, \Sigma p_{0i}x_i)} = \frac{\alpha_i}{L(\Sigma p_{1i}x_i, \Sigma p_{0i}x_i)} = \frac{\alpha_i}{\Omega}.$$

M12: Proof of Rule 4⊗Rule 2 (Appendix A.4)

$$\sum \beta_i \log\left(\frac{p_{1i}}{p_{0i}}\right) \Rightarrow (\Sigma\beta_i) \log\left(\frac{\Sigma p_{1i}x_i}{\Sigma p_{0i}x_i}\right) \Rightarrow (\Sigma\beta_i) \sum \alpha_i \log\left(\frac{p_{1i}}{p_{0i}}\right).$$

where

$$x_i = \frac{\delta_i}{L(p_{1i},p_{0i}B)}, \delta_i = \frac{\beta_i}{\Sigma\beta_i}, B = \frac{\Sigma p_{1i}x_i}{\Sigma p_{0i}x_i}, \alpha_i = \frac{L(v_{1i},v_{0i})}{\Sigma L(v_{1i},v_{0i})},$$

$$v_{1i} = \frac{p_{1i}x_i}{\Sigma p_{1i}x_i}, \text{ and } v_{0i} = \frac{p_{0i}x_i}{\Sigma p_{0i}x_i}.$$

Here, $x_i L(p_{1i},p_{0i}B) = \delta_i = \beta_i/\Sigma\beta_i$ and $v_{1i}/v_{0i} = p_{1i}/(p_{0i}B)$. Thus

$$\begin{aligned} L(v_{1i},v_{0i}) &= v_{0i}L(v_{1i}/v_{0i}, 1) = (v_{0i}/(p_{0i}B))L(p_{1i},p_{0i}B) \\ &= (x_i/(\Sigma p_{1i}x_i))L(p_{1i},p_{0i}B) = \delta_i/(\Sigma p_{1i}x_i) = (\beta_i/\Sigma\beta_i)(1/(\Sigma p_{1i}x_i)). \end{aligned}$$

$\therefore\ \alpha_i = \beta_i/\Sigma\beta_i$.